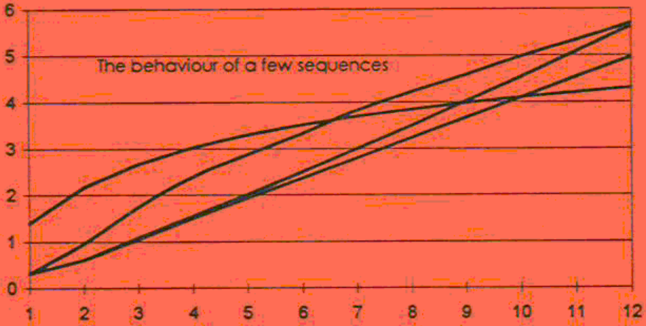
Computer Analysis of Number Sequences

Henry Ibstedt



American Research Press
Lupton, USA
1998

Computer Analysis of Number Sequences

Henry Ibstedt

Glimminge 2936
280 60 Broby
Sweden
(May-October)

7, rue du Sergent Blandan
92130 Issy les Moulineaux
France
(November-April)

E-mail: HIbstedt@swipnet.se

American Research Press
Lupton, USA
1998

The cover picture refers to the last article in this book. It illustrates how rapidly the sums $\Sigma\frac{1}{n}$, $\Sigma\frac{1}{n^{1.02}}$, $\Sigma\frac{1}{S(n)^2}$ and $\Sigma\frac{1}{S_r(n)^2}$, where S(n) is the Smarandache function and $S_r(n)$ is the sequences of square residues, grow beyond 1, 2, ...12. The y-axis scale is the natural logarithm of the number of terms required. A detailed discussion is given in chapter VI.

Referents:
Dr. R. Muller & J. Castillo, American Research Press
and
Prof. Mihaly Bencze, 6 Harmanului Street, 2212 Sacele 3, Jud. Brasov, Romania

Printed in the United States of America
by American Research Press
I 40 & Window Rock Road
Lupton, Box 199
AZ 86508, USA
E-mail: arp@cia-g.com

ISBN 1-879585-59-6

Standard Address Number 297-5092

Preface

This is a book on empirical number theory concentrating on the analysis of number sequences. Its focus is on a small part of a very large number of integer sequences defined by Florentin Smarandache. The author has, however, when appropriate included some other of his research results which organically belongs to this area. The content is organized into chapters according to the main considerations to make when programming the analysis. They are not mutually exclusive.

As in my previous book *Surfing on the ocean of numbers - a few Smarandache Notions and Similar Topics* an attempt has been made to present results so that they are easy to understand and not to burdensome to read. There are many tables, some of which may be used for reference, but in general they are there to show the overall results obtained at a glance. In most cases it is the way in which sequences behave, not the individual figures, which is of importance. Some graphs have been included to illustrate important findings.

Some of the results in this book were presented by the author at the *First International Conference on Smarandache Notions in Number Theory*, August 21-23, 1997, Craiova, Romania.

References have been given after some of the chapters. However, constant use has been made of the following Smarandache source materials:
Only Problems, Not Solutions!, *Florentin Smarandache.*
Some Notions and Questions in Number Theory, *C. Dumitrescu, V. Seleacu.*

Illustrations, graphics, layout and final editing up to camera ready form has been done by the author. Most tables have been created by direct transfer from computer files established at the time of computing to the manuscript so as to avoid typing errors. The results often involve very large numbers which are difficult to accommodate in pre-designed formats. It has therefore often been necessary to use several lines to represent a number in tables as well as text. All calulations have been carried out using *UBASIC, ver. 8.87.*

I am grateful to Dr. R. Muller who has given all possible help and encouragement during the work on this book. He and his colleagues at the *American Research Press* have at all times facilitated the work through rapid e-mail communications.

Finally, it is thanks to the patience and understanding of my wife Anne-Marie that this book has come about. But the winter months have passed, the book is finished and I am no longer going to be lost among manuscript pages. Summer is around the corner and the Swedish nature is waiting for us.

Paris, April 1998
Henry Ibstedt

Contents

Chapter I

Partition Sequences

I.1 Introduction

We will study the partition of the sequence of digits
of positive integers into two or more groups of digits whose corresponding integers may or may not possess a certain property.
<u>Example:</u>

17 | 7 | 101

is a partition of the 6-digit integer 177101 into three groups of digits (parts) each of which represents a prime, whereas

1 | 77 | 10 | 1

is a partition of the same integer into four parts none of which is a prime.

Obviously there are many ways in which and integer can be partitioned into groups of digits. The kind of partition described above may also be referred to as a de-concatenation in order to distinguish it from the classical concept of partition of an integer as first developed by Euler and later illustrated by Ferrer [1]. However, the kind of partition we are dealing with will be clear from the context. We will now use the classical partition of an integer as a means to help us formulate a strategy for partitioning sequences of integers in the sense defined above.
The partitions of 4 are: 1+1+1+1=2+1+1=2+2=3+1=4 with the corresponding Ferrer diagram:

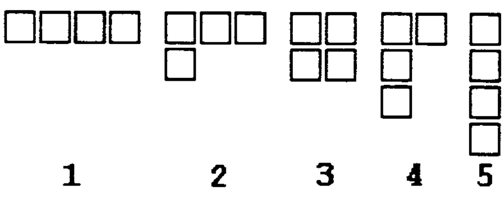


Diagram 1.The Ferrer diagram for 4.

The last partition corresponds to 4 itself which is of no interest to us. Furthermore, we will link the vertical representations to form the pattern shown in diagram 2:

1 2 3 4

Diagram 2. The Ferrer diagram for 4 rearranged and with partition numero 5 excluded.

In our applications we will also need to consider the order in which the partition elements occur. Diagram 2 shows the seven different ordered partitions possible for 4 and how they form a partition pattern for the partition of the 4-digit integer 9164 into groups of integers.

1 2 3 4 5 6 7

Diagram 3. The partition pattern for a 4-digit integer.

From this we see that the integer 9164 can be partitioned into perfect squares in exactly two ways, 9, 16, 4 (partition numero 3) and 9, 1, 64 (partition numero 4). This is how the mapping of and integer onto a partition pattern helps us study the properties of each partition element.

Only partitions 2 and 4 in diagram 2 contain partition elements of unequal sizes. These give rise to an increase of the number partition patterns through arrangement of the partition elements. It is easily understood that in a partition of 8 we have 2 parts of the type ▭ and 4 parts of the type □ then the number of arrangements will be $\frac{6!}{2!\cdot 4!}=15$. The integer 8 has 21 partitions of the type shown in diagram 2 giving rise to 127 ordered partitions. This could be calculated by considering each arrangement as we did above. However, we will soon arrive at this in a different way.

The above considerations will be useful in order to express the partitions of integers in explicit form. When we consider the partition of a very large number of integers to find out how many of them possess a certain property we will be able to use a different approach.

Let's represent an n-digit integer t in the form $t=a_1+a_2\cdot10+a_3\cdot10^2+ \ldots\ a_n\cdot10^{n-1}$. We introduce the following definitions and computer analysis related concepts in UBASIC.

> $\delta(t)=1$ if t possesses a certain property otherwise $\delta(t)=0$.
>
> \ symbolizes integer division, example 23678\100=23
>
> res symbolizes the remainder of the last performed integer division, in the example above res=678.

We will now take a look at the question of leading zeros. Is 3 | 07 a partition of 307 into two primes? Is 8 | 027 a partition of 8027 into two cubes? The author prefers to have a unique representation of an integer, 7 is a 1-digit integer written 7 not 07 and $3^3=27$ is written 27 - not 027. The computer, however, interprets 07 as 7 and 027 as 27. To avoid integers with leading zeros the function $\varepsilon(j,r)$ is defined as follows.

> $\varepsilon(j,r)=0$ if $r<10^{j-1}$ otherwise $\varepsilon(j,r)=1$, where r and j are context depending variables.

We will now use these functions to formulate an algorithm for calculation of the number of n-digit integers which can be partitioned so that the partition elements have a given property (being primes, squares, cubes, etc)

Lets denote the number of partitions of t by p(t).

<u>1-digit integers</u>: $1\le t\le 9$. Obviously there is nothing to partition, p(t)=0.

<u>2-digit integers</u>: $10\le t\le 99$. p(t) can only assume the values 0 and 1 and is given by

> $p(t)=\delta(t\backslash 10)\cdot\delta(res)$

3-digit integers: $100 \le t \le 999$. In the algorithm below use is made of the fact that we know p(t) for t<100.

$$p(t)=\delta(t\backslash 100)\cdot\varepsilon(1,res)\cdot(\delta(res)+p(t\backslash 100))+\delta(t\backslash 10)\cdot(\delta(res)+p(t\backslash 10)),$$

where we know that $p(t\backslash 10)=0$. It has been included in order to present a general algorithm for n-digit integers.

n-digit integers: $10^{n-1} \le t \le 10^{n}-1$.

$$p(t)=\sum_{k=1}^{n-1}\delta(t\backslash 10^{n-k})\varepsilon(n-k-1,res)(\delta(res)+p(res)) \tag{1}$$

We can now use (1) to calculate the number of partition patterns for n-digit integers. To do this we put $\delta(t\backslash n^{n-k})=1$ (the property of being a partition element), $\varepsilon(n\text{-}k\text{-}1,res)=1$ (a partition element may begin with a zero) and p(res)=p(k). This results in

$$p(n)=\sum_{k=1}^{n-1}(1+p(k))\text{, where } p(1)=0$$

Evaluation of the recursion formula gives $p(n)=2^{n-1}-1$ resulting in the following table of partition patterns.

Table 1. Partition patterns, N = number of patterns.

n-digit integer	1	2	3	4	5	6	7	8
N	0	1	3	7	15	31	63	127

The algorithm has been implemented in UBASIC on a Pentium 100 Mhz portable computer to examine various partition sequences.

I.2 The Smarandache Prime-Partial Digital Sequence

Definition: The prime-partial-digital sequence is the sequence of prime numbers with the property that each member of the sequence can be partitioned into groups of digits such that each group is a prime.

Table 2 shows the first one hundred terms of this sequence.

We will see in Chapter II.4 that the Smarandache Prime-Digital Sub-Sequence is infinite. The Prime-Partial Digital Sequence is in fact a subset of the Prime-Partial Digital Sub-Sequence and is therefore a fortiori infinite which settles affirmatively a conjecture by Sylvester Smith. [2]. We will make observations on the number of ways in which prime integers can be partitioned into primes. How close is the maximum number of ways this can be done to the theoretical maximum given in table 1 and what do these partitions look like?

Table 2. The first one hundred members of the prime-partial-digit sequence.

23	37	53	73	113	137	173	193	197	211
223	227	229	233	241	257	271	277	283	293
311	313	317	331	337	347	353	359	367	373
379	383	389	397	433	523	541	547	557	571
577	593	613	617	673	677	719	727	733	743
757	761	773	797	977	1013	1033	1093	1097	1117
1123	1129	1153	1171	1277	1319	1327	1361	1367	1373
1493	1637	1723	1733	1741	1747	1753	1759	1777	1783
1789	1913	1931	1933	1973	1979	1993	1997	2113	2131
2137	2179	2213	2237	2239	2243	2251	2267	2269	2273

The property that we will examine with the function $\delta(t)$ is whether t is a prime or not. In UBASIC this is done in the following program.

$\delta(t)$:

```
10 .D(t)                              '.D(t) is equivalent to δ(t)
20 if t=0 then z=0 :goto 40
30 if nxtprm(t-1)=t then z=1 else z=0
40 return(z)                          'z is the value of D(t)
```

The function ε(j,r) takes the following form in UBASIC.

```
ε(j,r):    10 .L(j%,r)                       '.L(j%,r) is equivalent to ε(j,r)
           20 if r<10^(j%-1) then y=0 else y=1
           30 return(y)                      'y is the value of ε(j,r)
```

These two functions were made work together with the algorithm (1) in a UBASIC program to produce the results shown in tables 3 and 4.

Table 3. The ratio m/q where m is the number of n-digit primes which can be partitioned into primes and q is the number of n-digit primes.

	n=2	n=3	n=4	n=5	n=6	n=7	n=8
m	4	51	383	3319	27111	234229	2040170
q	21	143	1061	8363	68906	586081	5096876
m/q	0.1905	0.3566	0.3610	0.3969	0.3934	0.3997	0.4003

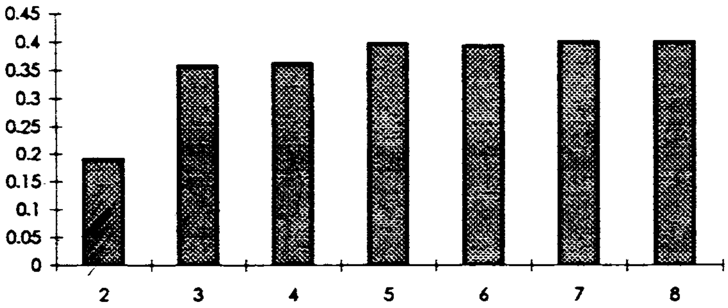


Diagram 4. The ratio m/q for n-digit integers (n=2,3,...8).

Table 4. The number of n-digit primes that can be partitioned into primes in p different ways. (The tail for n=8 is interposed.)

p	n=2	n=3	n=4	n=5	n=6	n=7	n=8
0	17	92	678	5044	41795	351852	3056706
1	4	40	251	1986	14789	118379	972095
2		10	83	769	6306	54102	436009
3		1	40	309	2811	25892	228428
4			5	135	1445	14205	139755
5			4	67	758	7676	82265
6				33	424	5012	55793
7				15	250	2922	35326
8				3	131	1877	24401
9				1	78	1258	17131
10				0	46	873	12424
11				0	22	550	8724
12				1	27	481	6671
13					12	277	4841
14					6	212	3573
15					3	147	2781
16					0	115	2052
17					0	62	1574
18					3	59	1244
19						33	992
20						34	811
21						18	617
22						12	500
23						6	417
24						10	311
25						4	264
26						4	214
27						2	154
28						3	150
29						0	94
30						1	83
31						2	85
32						0	65
33						0	51
34						0	42
35						0	51
36						0	30
37						1	32

p	n=8
38	*16*
39	*17*
40	*8*
41	*13*
42	*5*
43	*4*
44	*8*
45	*14*
46	*3*
47	*9*
48	*6*
49	*2*
50	*2*
51	*4*
52	*1*
53	*0*
54	*4*
55	*0*
56	*3*
57	*0*
58	*0*
59	*0*
60	*1*

Table 5. All possible partitions of 23733737 into primes.

1	2	3	7	3	3	7	3	7
2	23	7	3	3	7	3	7	
3	2	37	3	3	7	3	7	
4	2	3	73	3	7	3	7	
5	2	3	7	3	37	3	7	
6	2	3	7	3	3	73	7	
7	2	3	7	3	3	7	37	
8	23	73	3	7	3	7		
9	23	7	3	37	3	7		
10	23	7	3	3	73	7		
11	23	7	3	3	7	37		
12	2	37	3	37	3	7		
13	2	37	3	3	73	7		
14	2	37	3	3	7	37		
15	2	3	73	37	3	7		
16	2	3	73	3	73	7		
17	2	3	73	3	7	37		
18	2	3	7	3	37	37		
19	2	373	3	7	3	7		
20	2	3	733	7	3	7		
21	2	3	7	337	3	7		
22	2	3	7	3	373	7		
23	23	73	37	3	7			
24	23	73	3	23	7			
25	23	73	3	7	37			
26	23	7	3	37	37			
27	2	37	3	37	37			
28	2	3	73	37	37			
29	2	3733	7	3	7			
30	2	3	7	3373	7			
31	2	373	37	3	7			
32	2	373	3	73	7			
33	2	373	3	7	37			
34	2	373	3	7	37			
35	2	3	733	73	7			
36	2	3	733	7	37			
37	2	3	7	337	37			
38	23	733	7	3	7			
39	23	7	337	3	7			
40	23	7	3	373	7			
41	2	37	337	3	7			

Table 5. Continued.

42	2	37	3	373	7
43	2	3	73	373	7
44	2	373	373	7	
45	2	37337	3	7	
46	2	337	37	37	
47	23	733	73	7	
48	23	733	7	37	
49	23	7	337	37	
50	2	37	337	37	
51	23	73	373	7	
52	2	3733	73	7	
53	2	3733	7	37	
54	23	7	3373	7	
55	2	37	3373	7	
56	2	37337	37		
57	23	73373	7		
58	23	7	33737		
59	2	37	33737		
60	2373373	7			

With a slight exception for 6-digit integers the ratio between primes that can be partitioned into primes and the number of primes is increasing and indicates strongly that the prime-partial digital sequences is infinite.

To list the actual partitions would be impossible but for each n-digit integer sequence of primes there are a few primes that are record holders in the number of ways in which they can be partitioned into primes (p ways). Here is a list of those primes.

```
n=3      p=3    373
n=4      p=5    3137, 3373, 3733 and 3797
n=5      p=12   37337
n=6      p=18   237373, 537373 and 733373
n=7      p=37   2373373
n=8      p=60   23733737
```

We see that of the prime digits 2,3,5 and 7 the primes 3 and 7 play a dominant role in the composition of those primes that lend themselves to the maximum number of partitions into primes. All partitions can be displayed in

explicit form by executing a UBASIC program based on the partition patterns which were subject to a detailed discussion in the introduction. Table 5 shows the partitions for 23733737.

I.3 The Smarandache Square-Partial Digital Sequence

Definition: The square-partial digital sequence is the sequence of perfect squares which can be partitioned into two or more groups of digits which are also perfect squares.

The first one hundred terms of this sequence is shown in table 6.

Table 6. The first one hundred terms f the square-partial digital sequence.

49	100	144	169	361	400	441	900	1225	1369
1444	1600	1681	1936	2500	3249	3600	4225	4900	6400
8100	9025	9409	10000	10404	11025	11449	11664	12100	12544
14161	14400	14641	15625	16641	16900	19044	19600	22500	25600
28900	32400	36100	36481	40000	40401	41616	42025	43681	44100
44944	48400	49729	52900	57600	62500	64009	67600	72900	78400
81225	84100	90000	93025	93636	96100	99225	102400	105625	108900
115600	116964	117649	119025	121104	122500	129600	136161	136900	140625
143641	144400	152100	157609	160000	161604	164025	166464	168100	170569
176400	184900	193600	194481	202500	211600	220900	225625	230400	237169

This sequence is infinite. If all the infinitely many squares of the form $s\cdot 10^{2k}$, where s is a perfect square, are removed the sequence is still infinite. Charles Ashbacher [3] has proved this in the case where leading zeros are accepted. In order to continue to use unique representation of integers the proof has to be changed.

Proof: There are infinitely many squares of the form $(10^{2k+1}+2)^2$. In expanded form these can be written $(10^k)^2\cdot 10^{2(k+1)}+(2\cdot 10^k)^2\cdot 10+2^2$. From this we see that $(10^{2k+1}+2)^2$ can be partitioned into the three squares: 10^{2k}, $4\cdot 10^{2k}$ and 4.

The methods and programs developed in the introduction to this chapter have been used in the analysis of this sequence with the only change that we study a different partition property defined though the function δ(t) which now takes the form:

```
10 .D(t)
20 if (isqrt(t))^2=t then z=1 else z=0
30 return(z)
```

Use has been made of the Ubasic function isqrt(x) which gives the integer part of the square root of x.

The analysis was carried out for 2, 3, ... 8 digit squares. The results are presented in the form used in the previous section.

Table 7. The ratio m/q. m is the number of squares which can be partitioned into squares and q is the number of n-digit squares.

	n=2	n=3	n=4	n=5	n=6	n=7	n=8
m	1	7	15	44	114	316	883
q	6	15	68	217	683	2163	6837
m/q	0.1667	0.4667	0.2206	0.2028	0.1669	0.1461	0.1292

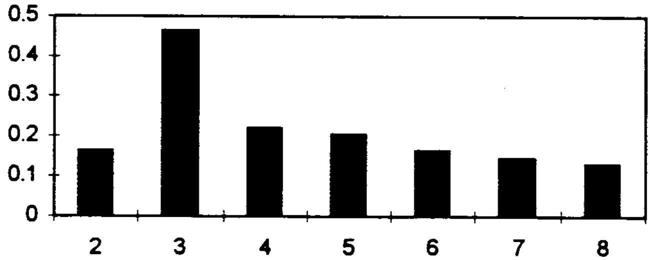


Diagram 5. The ratio m/q for n-digit integers (n = 1,2 ... 8).

For squares with more than three digits the graph shows an interesting decrease as if it were approaching an horizontal asymptote. Is this the case? For each n-digit integer sequence of squares we list those squares which can be partitioned into squares in a maximum number of ways (p ways):

```
n=3      p=1    100, 144, 169, 361, 400, 441, 900
n=4      p=3    4900
n=5      p=5    11449, 14400, 16900, 36100, 44100
n=6      p=5    144400, 490000
n=7      p=7    1144900
n=8      p=8    14440000, 36144144, 49491225
```

This list is dominated by squares ending on an even number of zeros. The digit 4 occurs most frequently while the digits 7 and 8 are lacking. This section is terminated by a table of maximum partitions for n=8.

Table 8a. All possible partitions of 49491225 into squares.

1	4	9	4	9	1	225
2	4	9	4	9	1225	
3	49	4	9	1	225	
4	4	9	49	1	225	
5	49	49	1	225		
6	49	4	9	1225		
7	4	9	49	1225		
8	49	49	1225			

Table 8b. All possible partitions of 36144144 into squares.

1	36	1	4	4	1	4	4
2	361	4	4	1	4	4	
3	36	144	1	4	4		
4	36	1	441	4	4		
5	36	1	4	4	144		
6	361	441	4	4			
7	361	4	4	144			
8	36	144	144				

Table 8c. All possible partitions of 144440000 into squares.

1	1	4	4	4	0	0	0	0
2	144	4	0	0	0	0		
3	1	4	4	400	0	0		
4	1444	0	0	0	0			
5	144	400	0	0				
6	1	4	4	40000				
7	144400	0	0					
8	144	40000						

I.4 The Smarandache Cube-Partial Digital Sequence

Definition: The cube-partial digital sequence is the sequence of cubes which can be partitioned into two or more groups of digits which are also cubes.

It is obvious that all integers of the form $n^3 \cdot 10^{3k}$ where n is an integer belong to the sequence, but if we let n be an integer whose last digit $\neq 0$, then there are only two cubes $m=n^3<10^9$ which can be partitioned into cubes. They are:

$22^3=10648$ with the partition 1, 0, 64, 8,
and
$303^3=27818127$ with the partition 27, 8, 1, 8, 1, 27.

In spite of this the sequence has infinitely many members with the last digit $\neq 0$. This is due to the fact that all cubes of the form $(3 \cdot 10^{3k+2}+3)^3$ can be written in the form $27 \cdot 10^{9k+6}+81 \cdot 10^{6k+4}+81 \cdot 10^{3k+2}+27$ which gives the partition:

27, 0, 0 ...(3k zeros), 81, 0 ,0 ...(3k zeros), 81, 0 ,0 ... (3kzeros), 27.

The integer 303 can be looked at as the generator of a whole family of members of the cube-partial digital sequence.

Question: Is 303 the only generator that guaranties that there are infinitely many members of this sequence with non-zero last digits?

L5 Partition of {1, 2, 3, ... 2n} into trigrades using Pythagorean triples.

The relation $2^k+3^k+7^k=1^k+5^k+6^k$ holds for k=1,2. It is called a bigrade and is written $2,3,7 \overset{2}{=} 1,5,6$. To find similar relations which hold for a larger range of values for k has attracted a lot of interest [4]. Here is an example of a pentagrade

$$1,5,10,18,23,27 \overset{5}{=} 2,3,13,15,25,26$$

Definition: A relation

$$a_1^m+a_2^m+ \dots a_n^m= b_1^m+b_2^m+ \dots b_n^m$$

which holds for m=1, 2, ... k is called a k-grade. In abbreviated form this is written

$$a_1,a_2,\dots a_n \overset{k}{=} b_1,b_2,\dots b_n$$

The purpose of this study is to give an explicit expression for the partition of S={1,2, .. 2n} into a={ a_1, a_2,...,a_n} and B={ b_1, b_2,...,b_n} so that $A \overset{3}{=} B$ [1]. We will see that this partition can be obtained through a suitable choice of Pythagorean numbers. The trigrades generated in this way will in general have a large number of terms. They can, however, be used to produce trigrades with only eight terms:

$$a_1,a_2,a_3,a_4 \overset{3}{=} b_1,b_2,b_3,b_4$$

Let A_0 consist of all odd integers x in the interval $1 \le x \le n-1$, where n is assumed to be even, and all even integers in the interval $n+2 \le x \le 2n$. $B_0=A'_0$
Consider

$$S_m = \frac{1}{2}\sum_{k=1}^{2n} k^m \quad \text{(Half the sum of m}^{\text{th}}\text{ powers of S)}$$

$$A_m = \sum_{k=1}^{n/2}(2k-1)^m + \sum_{k=1}^{n/2}(n+2k)^m$$

Evaluating these sums result in

---

[1] S = A ∪ B with A ∩ B = Ø, b = a' (a' stands for the complement of a).

$$A_1 - S_1 = 0 \text{ (i.e. } A_0 \overset{1}{=} B_0) \tag{1}$$

$$A_2 - S_2 = n^2/2 \tag{2}$$

$$A_3 - S_3 = 3n^2(2n+1)/4 \tag{3}$$

A solution is attempted by constructing new partitions A and B through exchange of elements between A_0 and B_0 so that (1) is maintained while satisfying (2) and (3).

Exchange: $\{x_1, x_2\} \varepsilon B_0 \rightarrow A$ and $\{y_1, y_2\} \varepsilon A_0 \rightarrow B$ so that

$$y_1 + y_2 = x_1 + x_2 \tag{1.1}$$

$$(y_1^2 + y_2^2) - (x_1^2 + x_2^2) = n^2/2 \tag{2.1}$$

$$(y_1^3 + y_2^3) - (x_1^3 + x_2^3) = 3n^2(2n+1)/4 \tag{3.1}$$

Assume $y_2 > y_1$ and write (1.1) as:

$$x_1 = y_1 + d; \qquad x_2 = y_2 - d \tag{1.2}$$

Substituting this and simplifying (2.1) and (3.1) results in:

$$2d(y_2 - y_1 - d) = n^2/2 \tag{2.2}$$

$$y_1 + y_2 = 2n + 1 \tag{3.2}$$

Assume $j \varepsilon \{1, 2, \dots n/2\}$ and write (3.2) as

$$y_2 = n + 2j; \qquad y_1 = n + 1 - 2j \tag{3.3}$$

Insert these expressions in (2.2)

$$n^2 = 4d(4j - d - 1) \tag{2.3}$$

Integer solutions for d require the discriminant of this equation to be a square z^2.

$$(4j - 1)^2 - n^2 = z^2 \text{ with } n/4 + 1 \le j \le n/2 \qquad (4)$$

Let a, b and c be a Pythagorean triple with $2 \mid b$ and let k be an odd integer, (4) is then satisfied by $4j-1=kc$, $n=kb$ and $z=ka$. It remains to evaluate x_1, x_2, y_1 and y_2 in terms of a, b, c and k. To begin with we have $j=(kc+1)/4$. Since $j \in Z_+$ and $c\equiv1 \pmod 4$ we can write $k=4e-1$ where $e \in Z_+$. This gives $d=(4e-1)(c \pm a)$ and the final solution:

$$x_{1,2} = ((4e - 1)(2b \pm a) + 1)/2 \qquad (5)$$

$$y_{1,2} = ((4e - 1)(2b \pm c) + 1)/2 \qquad (6)$$

The conditions $2b>a$ and $2b>c$ restrict the number of Pythagorean triples which can be used to generate trigrade partitions of the first N positive integers where N is given by:

$$N = 2b(4e - 1), \quad e \in Z_+ \qquad (7)$$

A further condition is that x_1 and x_2 given by (5) must belong to A_0, i.e. $x\equiv0 \pmod 2$ if $0<x\le n$ and $x\equiv1 \pmod 2$ if $n<x\le 2n$. This is met if $2b+a\equiv-1 \pmod 4$. The corresponding condition must be met for y_1 and y_2.
From the above conditions and (7) we see that a Pythagorean triple generates zero or infintely many trigrades. If it produces infinitely many trigrades then the number of terms N of these are in arithmetic progression with first term 6b and difference 8b. The first non-trivial Pythagorean triples are listed in table 9. Those which can be used to produce trigrades are marked in column N by giving the number of terms in the trigrade which has the least number of terms.

The trigrade with the smallest number of terms generated in this way is (the exchange terms are underscored);

$$1, 2, 7, \underline{8}, 9, 11, 14, 16, \underline{17}, 18, 22, 24 \stackrel{3}{=} 2, 4, \underline{5}, 6, 10, 12, 13, 15, 19, \underline{20}, 21, 23$$

Table 9. Pythagorean triples and their corresponding smallest N.

a	b	c	N	a	b	c	N	a	b	c	N
3	4	5	24	55	48	73	288	17	144	145	
5	12	13		77	36	85		51	140	149	840
15	8	17		13	84	85		85	132	157	
7	24	25	144	39	80	89	480	119	120	169	720
21	20	29		65	72	97		165	52	173	
35	12	37		99	20	101		19	180	181	1080
9	40	41		91	60	109	360	57	176	185	
45	28	53		15	112	113	672	153	104	185	
11	60	61	360	117	44	125		95	168	193	1008
33	56	65		105	88	137		195	28	197	
63	16	65		143	24	145		133	156	205	

Since the sets A_0 and B_0 are defined from the outset the trigrade partitions can be unambigously described in terms of these sets and the exchange terms as shown in table 10. From this table it is seen that the trigrades corresponding to the last two Pythagorean triples have the same number of terms. This means that we can obtain trigrades with only eight terms by subtraction. For example we see from table 10 that

$$197, 164, 344, 17 \stackrel{3}{=} 317, 44, 272, 89 \tag{8}$$

A k-grade $a_1,a_2,\ldots a_n \stackrel{k}{=} b_1,b_2,\ldots b_n$ is invariant under translation. To prove this consider the sets $A=\{x+a_1, x+a_2,\ldots,x+a_n\}$ and $B=\{x+b_1, x+b_2,\ldots,x+b_n\}$. Expand each term to the power m where $m\leq k$.

$$\sum_{i=1}^{n}(x+a_i)^m = \sum_{i=1}^{n}\sum_{j=0}^{m}\binom{m}{j}x^j a_i^{m-j} = \sum_{j=0}^{m}\binom{m}{j}x^j\sum_{i=1}^{n}a_i^{m-j} = \sum_{j=0}^{m}\binom{m}{j}x^j\sum_{i=1}^{n}b_i^{m-j} = \sum_{i=1}^{n}\sum_{j=0}^{m}\binom{m}{j}x^j b_i^{m-j} = \sum_{i=1}^{n}(x+b_i)^m$$

where the assumed k-grade property has been used in exchanging a_i and b_i. We can consequently reduce all terms in the above trigrade by 17 (if we reduced by 16 instead the eight terms would be positive and realively prime but would prevent further reductions) to obtain

$$0, 147, 180, 327 \stackrel{3}{=} 27, 72, 255, 300 \tag{8.1}$$

The k-grade property is evidently invariant under division by a common factor. Hence

$$0, 49, 60, 109 \stackrel{3}{=} 9, 24, 85, 100 \tag{8.2}$$

The steps shown above carried out for Pythagorean triples c<1000 result in quite a large number of eight-term trigrades as is shown in table 11. To reduce the trigrades so that the smallest term is 0 and divided the terms with their LCM proved essential in order to avoid duplications. When implementing the algorithms on a computer several solutions were obtained which after reduction proved to be identical.

Table 10. Description of trigrades

#	Triple N	a and B
1	3,4,5 24	a={1,3,...,11}∪{14,16,...,24}∪{8,17}-{5,20} B={2,4,...,12}∪{13,15,...,23}∪{5,20}-{8,17}
2	56	A={1,3,...,27}∪{30,32,...,56}∪{18,39}-{11,46} B={2,4,...,28}∪{29,31,...,55}∪{11,46}-{18,39}
3	88	A={1,3,...,43}∪{46,48,...,88}∪{28,61}-{17,72} B={2,4,...,44}∪{45,47,...,87}∪{17,72}-{28,61}
4	7,24,25 144	A={1,3,...,71}∪{74,76,...,144}∪{62,83}-{35,110} B={2,4,...,72}∪{73,75,...,143}∪{35,110}-{62,83}
5	336	A={1,3,...,167}∪{170,172,...,336}∪{144,193}-{81,256} B={2,4,...,168}∪{169,171,...,335}∪{81,256}-{144,193}
6	528	A={1,3,...,263}∪{266,268,...,528}∪{226,303}-{127,402} B={2,4,...,264}∪{265,267,...,527}∪{127,402}-{226,303}
7	55,48,73 288	A={1,3,...,143}∪{146,148,...,288}∪{62,227}-{35,254} B={2,4,...,144}∪{145,147,...,287}∪{35,254}-{62,227}
8	672	A={1,3,...,335}∪{338,340,...,672}∪{144,529}-{81,592} B={2,4,...,336}∪{337,339,...,671}∪{81,592}-{144,529}
9	1056	A={1,3,...,527}∪{530,532,...,1056}∪{226,831}-{127,930} B={2,4,...,528}∪{529,531,...,1055}∪{127,930}-{226,831}

Table 10. Description of trigrades, continued.

#	Triple N	a and B
10	11,60,61 360	A={1,3,...,179}∪{182,184,...,360}∪{164,197}-{89,272} B={2,4,...,180}∪{181,183,...,359}∪{89,272}-{164,197}

11	840	$A=\{1,3,\ldots,419\}\cup\{422,424,\ldots,840\}\cup\{382,459\}-\{207,634\}$ $B=\{2,4,\ldots,420\}\cup\{421,423,\ldots,839\}\cup\{207,634\}-\{382,459\}$
12	91,60,109 360	$A=\{1,3,\ldots,179\}\cup\{182,184,\ldots,360\}\cup\{44,317\}-\{17,344\}$ $B=\{2,4,\ldots,180\}\cup\{181,183,\ldots,359\cup\{17,344\}-\{44,317\}$
12	840	$A=\{1,3,\ldots,419\}\cup\{422,424,\ldots,840\}\cup\{102,739\}-\{39,802\}$ $B=\{2,4,\ldots,420\}\cup\{421,423,\ldots,839\}\cup\{39,802\}-\{102,739\}$

Table 11. Trigrades

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0	49	60	109	9	24	85	100
0	67	102	169	22	25	144	147
0	85	136	221	25	36	185	196
0	103	122	225	22	45	180	203
0	107	242	349	25	62	287	324
0	125	638	763	50	63	700	713
0	129	244	373	48	49	324	325
0	151	382	533	49	74	459	484
0	155	230	385	34	77	308	351
0	165	184	349	25	84	265	324
0	181	384	565	60	81	484	505
0	185	262	447	10	147	300	437
0	203	554	757	81	86	671	676
0	215	370	585	46	117	468	539
0	229	304	533	49	108	425	484
0	233	278	511	63	86	425	448
0	241	556	797	72	121	676	725
0	271	678	949	102	117	832	847
0	283	542	825	58	165	660	767
0	293	370	663	75	118	545	588
0	301	456	757	81	132	625	676
0	305	458	763	63	158	605	700
0	331	994	1325	110	169	1156	1215
0	343	1074	1417	117	174	1243	1300

Table 11. Trigrades, continued.

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0	349	376	725	49	180	545	676
0	359	506	865	81	170	695	784
0	365	400	765	76	153	612	689
0	379	522	901	54	225	676	847
0	381	640	1021	121	156	865	900

0	385	996	1381	96	225	1156	1285
0	407	1262	1669	122	225	1444	1547
0	461	1392	1853	153	236	1617	1700
0	469	1264	1733	108	289	1444	1625
0	477	764	1241	153	188	1053	1088
0	501	688	1189	76	289	900	1113
0	511	1348	1859	148	275	1584	1711
0	541	736	1277	121	252	1025	1156
0	551	796	1347	147	236	1111	1200
0	593	824	1417	121	296	1121	1296
0	631	1138	1769	169	306	1463	1600
0	649	964	1613	169	288	1325	1444
0	787	838	1625	162	325	1300	1463
0	817	1048	1865	245	292	1573	1620
0	835	1880	2715	296	363	2352	2419
0	845	2522	3367	170	567	2800	3197
0	849	1084	1933	169	408	1525	1764
0	925	2190	3115	34	847	2268	3081

I.6 Smarandache Non-Null Squares.

Question: In how many ways can n be written as a sum of non-null squares, disregarding the order of the terms?

Example: $9=1^2+1^2+1^2+1^2+1^2+1^2+1^2+1^2+1^2$
$=1^2+1^2+1^2+1^2+1^2+2^2$
$=1^2+2^2+2^2$
$=3^2$

Let us denote the number of ways by $\eta(n)$. The example shows that $\eta(9)=4$. Obviously the non-null squares representation which has the largest number of terms is a representation by n squares of 1 and the representation with the smallest number of terms is a representation where we use the largest possible squares. If n is a perfect square then $(\sqrt{n})^2$ is one-term representation.

Let's assume that we know $\eta(x)$ for all integers $x \le n-1$. Furthermore we make the definition:

$$\delta(n) = \begin{cases} 1 \text{ if n is a square} \\ 0 \text{ otherwise} \end{cases}$$

With these precisions we can write the algorithm for calculation of $\eta(n)$ in the following way:

$$\eta(n) = 1 + \delta(n) + \sum_{i=2}^{[\sqrt{n}]-\delta(n)} \eta(n - i^2), \quad \text{with } \eta(1) = 1$$

This algorithm was implemented for $n \leq 100$. As is shown in table 12 this sequence, after a slow start, grows very rapidly.

Table 12. The Non-Null Square Representations.

1	1	1	2	2	2	2	3	4	4
4	5	7	7	7	9	12	13	13	16
20	23	23	27	35	41	42	47	61	71
75	82	104	124	134	146	178	217	237	258
307	377	419	456	535	651	739	804	933	1126
1300	1422	1629	1955	2275	2513	2846	3397	3972	4435
4990	5904	6933	7807	8766	10268	12097	13718	15409	17895
21087	24076	27076	31248	36736	42214	47568	54636	64017	73924
83554	95596	111637	129306	146714	167379	194807	226021	257447	293255
340106	394953	451408	514025	594103	690035	790967	901118	1038450	1205451

L7 Smarandache Non-Null Cubes.

In how many ways $\tau(n)$ can n be written as a sum of non-null cubes? In this case there are much fewer representations. For non-null squares we have $\eta(9)=4$ whereas for non-null cubes we have $\tau(9)=2$. Calculation of the first 100 terms follows the same method as for the non-null squares with corresponding changes in our algorithms.

$$\delta(n) = \begin{cases} 1 \text{ if } n \text{ is a cube} \\ 0 \text{ otherwise} \end{cases}$$

$$\tau(n) = 1 + \delta(n) + \sum_{i=2}^{[\sqrt[3]{n}]-\delta(n)} \eta(n - i^3), \quad \text{with } \tau(1) = 1$$

Table 13. The Non-Null Cube Representations.

1	1	1	1	1	1	1	2	2	2
2	2	2	2	2	3	3	3	3	3
3	3	3	4	4	4	5	5	5	5
5	6	6	6	8	8	8	8	8	9
9	9	12	12	12	12	12	13	13	13
17	17	17	18	18	19	19	19	24	24
24	27	27	29	29	29	35	35	35	41
41	44	44	44	51	51	51	61	61	65
66	66	74	74	74	89	89	94	98	98
109	109	109	130	130	136	146	146	162	162

References

[1] Hardy and Wright, An Introduction to the Theory of Numbers, *Oxford University Press.*

[2] Sylvester Smith, A Set of Conjectures on Smarandache Sequences, *Bulletin of Pure and Applied Sciences, Vol. 15, 1996.*

[3] Charles Ashbacher, Collection of Problems on Smarandache Notions, *Erhus University Press, Vail, 1996.*

[4] Albert H. Beiler, Recreations in the Theory of Numbers, *Dover Publications, Inc. New York.*

Chapter II

Recursive Integer Sequences

II. 1 The Non-Arithmetic Progression

We consider an ascending sequence of positive integers $a_1, a_2, \ldots a_n$ such that each element is as small as possible and no t-term arithmetic progression is in the sequence. In order to attack the problem of building such sequences we need a more operational definition.

Definition: The t-term non-arithmetic progression is defined as the set : $\{a_i : a_i$ is the smallest integer such that $a_i > a_{i-1}$ and such that for $k \leq i$ there are at most t-1 equal differences $a_k - a_{k_1} = a_{k_1} - a_{k_2} = \ldots = a_{k_{t-2}} - a_{k_{t-1}} \}$

From this definition we can easily formulate the starting set of a t-term non-arithmetic progression:

$$\{1, 2, 3 \ldots\ldots t-1, t+1\} \text{ or } \{a_i : a_i = i \text{ for } i \leq t-1 \text{ and } a_t = t+1 \text{ where } t \geq 3\}$$

It may seem clumsy to bother to express these simple definitions in stringent terms but it is in fact necessary in order to formulate a computer algorithm to generate the terms of these sequences.

Question: How does the density of a t-term non-arithmetic progression vary with t, i.e. how does the fraction a_k/k behave for $t \geq 3$?

Strategy for building a t-term non-arithmetic progression: Given the terms $a_1, a_2, \ldots a_k$ we will examine in turn the following candidates for the term a_{k+1}:

$$a_{k+1} = a_k + d,\ d=1, 2, 3, \ldots$$

Our solution is the smallest d for which none of the sets

$$\{a_1, a_2, \ldots a_k, a_k+d, a_k+d-e, a_k+d-2e, \ldots a_k+d-(t-1)\cdot e : e \geq d\}$$

contains a t-term arithmetic progression.

We are certain that a_{k+1} exists because in the worst case we may have to continue constructing sets until the term $a_k+d-(t-1)e$ is less than 1 in which case all possibilities have been tried with no t terms in arithmetic progression. The method is illustrated with an example in diagram 1.

In the computer application of the above method the known terms of a no t-term arithmetic progression were stored in an array. The trial terms were in each case added to this array. In the example we have for d=1, e=1 the array: 1,2,3,5,6,8,9,10,11,10,9,8. The terms are arranged in ascending order: 1,2,3,5,6,8,8,9,9,10,10,11. Three terms 8,9 and 10 are duplicated and 11 therefore has to be rejected. For d=3, e=3 we have 1,2,3,5,6,8,9,10,13,10,7,4 or in ascending order: 1,2,3,4,5,6,7,8,9,10,10,13 this is acceptable but we have to check for all values of e that produce terms which may form a 4-term arithmetic progression and as we can see from diagram 1 this happens for d=3, e=4, so 13 has to be rejected. However, for d=5, e=5 no 4-term arithmetic progression is formed and e=6 does not produce terms that need to be checked, hence $a_9 = 15$.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
Known terms		1	2	3		5	6		8	9	10						
Trials																	
d=1	e=1								8	9	10	11					reject 11
d=2	e=2						6		8		10		12				reject 12
d=3	e=3				4			7			10			13			try next e
	e=4	1				5				9				13			reject 13
d=4	e=3		2				6				10				14		reject 14
d=5	e=5					5					10					15	accept 15

Diagram 1. To find the 9th term of the 4-term non-arithmetic progression.

Routines for ordering an array in ascending order and checking for duplication of terms were included in a *QBASIC* program to implement the above strategy.

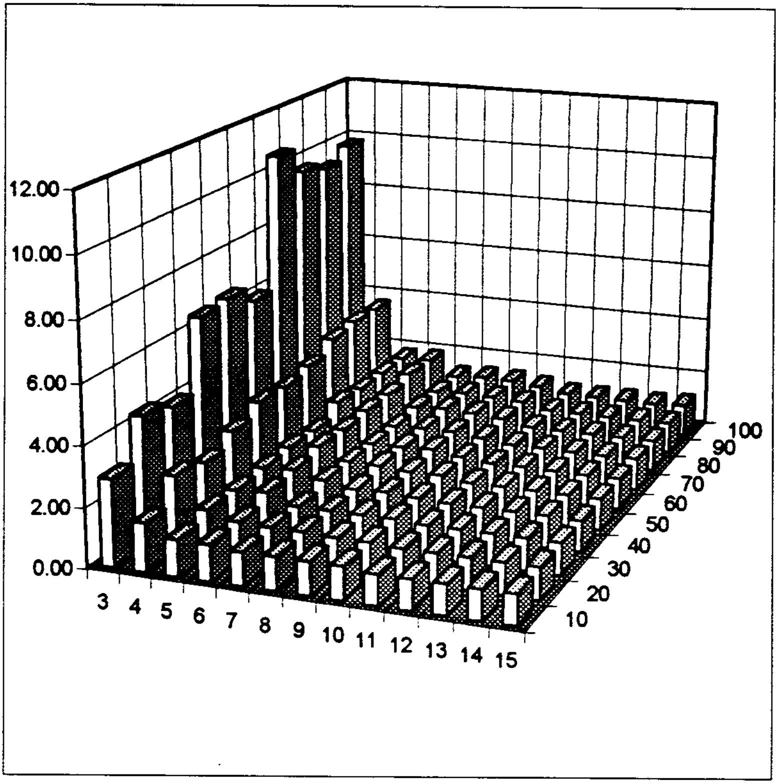


Diagram 2. a_k/k for non-arithmetic progressions with t=3, 4, 5, ... 15. Bars are shown for k = multiples of 10.

Results and observations: Calculations were carried out for $3 \le t \le 15$ to find the first 100 terms of each sequence. The first 65 terms and the 100^{th} term are shown in table 1. In diagram 2 the fractions a_k/k has been chosen as a measure of the density of these sequences. The looser the terms are packed the larger is a_k/k. In fact for t>100 the value of $a_k/k = 1$ for the first 100 terms.

In table 1 there is an interesting leap for t=3 between the 64th and the 65th terms in that a_{64} = 365 and a_{65} = 730. Looking a little closer at such leaps we find that:

Table 1. The 65 first terms of the non-arithmetic progressions for t=3 to 15.

#	t=3	t=4	t=5	t=6	t=7	t=8	t=9	t=10	t=11	t=12	t=13	t=14	t=15
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	4	3	3	3	3	3	3	3	3	3	3	3	3
4	5	5	4	4	4	4	4	4	4	4	4	4	4
5	10	6	6	5	5	5	5	5	5	5	5	5	5
6	11	8	7	7	6	6	6	6	6	6	6	6	6
7	13	9	8	8	8	7	7	7	7	7	7	7	7
8	14	10	9	9	9	9	8	8	8	8	8	8	8
9	28	15	11	10	10	10	10	9	9	9	9	9	9
10	29	16	12	12	11	11	11	11	10	10	10	10	10
11	31	17	13	13	12	12	12	12	12	11	11	11	11
12	32	19	14	14	13	13	13	13	13	13	12	12	12
13	37	26	16	15	15	14	14	14	14	14	14	13	13
14	38	27	17	17	16	16	15	15	15	15	15	15	14
15	40	29	18	18	17	17	16	16	16	16	16	16	16
16	41	30	19	19	18	18	17	17	17	17	17	17	17
17	82	31	26	20	19	19	19	18	18	18	18	18	18
18	83	34	27	22	20	20	20	20	19	19	19	19	19
19	85	37	28	23	22	21	21	21	20	20	20	20	20
20	86	49	29	24	23	23	22	22	21	21	21	21	21
21	91	50	31	25	24	24	23	23	23	22	22	22	22
22	92	51	32	26	25	25	24	24	24	24	23	23	23
23	94	53	33	33	26	26	27	25	25	25	24	24	24
24	95	54	34	34	27	27	28	26	26	26	25	25	25
25	109	56	36	35	29	28	29	27	27	27	27	26	26
26	110	57	37	36	30	30	30	28	28	28	28	28	27
27	112	58	38	37	31	31	31	31	29	29	29	29	28
28	113	63	39	39	32	32	32	32	30	30	30	30	29
29	118	65	41	43	33	33	33	33	31	31	31	31	31
30	119	66	42	44	34	34	34	34	32	32	32	32	32
31	121	67	43	45	36	35	37	35	34	33	33	33	33
32	122	80	44	46	37	37	38	36	35	35	34	34	34
33	244	87	51	47	38	38	39	37	36	36	35	35	35
34	245	88	52	49	39	39	40	38	37	37	36	36	36
35	247	89	53	50	40	40	41	39	38	38	37	37	37
36	248	91	54	51	41	41	43	41	39	39	38	38	38
37	253	94	56	52	50	42	44	42	40	40	40	39	39

Table continued on next page.

Table 1 continued.

#	t-3	t=4	t=5	t=6	t=7	t=8	t=9	t=10	t=11	t=12	t=13	t=14	t=15
38	254	99	57	59	51	44	45	43	41	41	41	41	40
39	256	102	58	60	52	45	46	44	42	42	42	42	41
40	257	105	59	62	53	46	47	45	43	43	43	43	42
41	271	106	61	63	54	47	48	49	45	44	44	44	45
42	272	109	62	64	55	48	49	50	46	46	45	45	46
43	274	110	63	65	57	49	50	51	47	47	46	46	47
44	275	111	64	66	58	50	53	52	48	48	47	47	48
45	280	122	66	68	59	59	55	53	49	49	48	48	49
46	281	126	67	69	60	60	56	54	50	50	49	49	50
47	283	136	68	71	61	61	57	55	51	51	50	50	51
48	284	145	69	73	62	62	58	58	52	52	51	51	52
49	325	149	76	77	64	63	59	59	53	53	53	52	53
50	326	151	77	85	65	64	60	60	54	54	54	54	54
51	328	152	78	87	66	65	64	61	56	55	55	55	55
52	329	160	79	88	67	67	65	62	57	57	56	56	56
53	334	163	81	89	68	69	66	63	58	58	57	57	58
54	335	167	82	90	69	70	67	64	59	59	58	58	59
55	337	169	83	91	71	71	68	65	60	60	59	59	60
56	338	170	84	93	72	72	69	66	61	61	60	60	61
57	352	171	86	96	73	74	70	68	62	62	61	61	62
58	353	174	87	97	74	75	71	69	63	63	62	62	63
59	355	176	88	98	75	76	78	70	64	64	63	63	64
60	356	177	89	99	76	77	79	71	65	65	64	64	65
61	361	183	91	100	78	78	80	72	67	66	66	65	66
62	362	187	92	103	79	79	81	73	68	68	67	67	67
63	364	188	93	104	80	81	82	74	69	69	68	68	68
64	365	194	94	107	81	84	83	75	70	70	69	69	69
65	730	196	126	111	82	85	84	77	71	71	70	70	70
...													
100	977	360	179	183	130	139	138	126	109	109	108	108	113

Leap starts at		Leap finishes at	
5		10	
14	$=3\cdot5-1$	28	$=2\cdot14$
41	$=3\cdot14-1$	82	$=2\cdot41$
122	$=3\cdot41-1$	244	$=2\cdot122$
365	$=3\cdot122-1$	730	$=2\cdot365$

Does this chain of regularity continue indefinitely?

Sometimes it is easier to look at what is missing than to look at what we have. Here are some observations on the only excluded integers when forming the first 100 terms for t=11, 12 , 13 and 14.

For t=11: 11, 22, 33, 44, 55, 66, 77, 88, 99 The *nth* missing integer is $11\cdot n$

For t=12: 12, 23, 34, 45, 56, 67, 78, 89, 100 The *nth* missing integer is $11\cdot n+1$

For t=13: 13, 26, 39, 52, 65, 78, 91, 104 The *nth* missing integer is $13\cdot n$

For t=14: 14, 27, 40, 53, 66, 79, 92, 105 The *nth* missing integer is $13\cdot n+1$

Do these regularities of missing integers continue indefinitely? What about similar observations for other values of t?

II.2 The Prime-Product Sequence

Definition: The terms of the prime-product sequence are defined through $\{t_n : t_n = p_n\#+1, p_n \text{ is the } nth \text{ prime number}\}$, where $p_n\#$ denotes the product of all prime numbers which are less than or equal to p_n.

The sequence begins {3, 7, 31, 211, 2311, 30031, ... }. In the initial definition of this sequence t_1 was defined to be equal to 2. However, there seems to be no reason for this exception.

Question: How many members of this sequence are prime numbers?

The question is in the same category as questions like *'How many prime twins are there?, How many Carmichael numbers are there?, etc.'* So we may have to contend ourselves by finding how frequently we find prime numbers when examining a fairly large number of terms of this sequence.

From the definition it is clear that the smallest prime number which divides t_n is larger than p_n. The terms of this sequence grow rapidly. The prime number functions *prmdiv(n)* and *nxtprm(n)* built into the *Ubasic* programming language were used to construct a prime factorization program for $n<10^{19}$. This program was used to factorize the 18 first terms of the sequence. An elliptic curve factorization program, ECM.UB, conceived by Y. Kida was adapted to generate and factorize further terms up to and including the 49*th* term. The result is shown in table 2. All terms analysed were found to be square free. A scatter diagram, Diagram 3, illustrates how many prime factors there are in each term .

The 50*th* term presented a problem. $t_{50}=126173\cdot n$, where n has at least two factors. At this point prime factorization begins to be too time consuming and after a few more terms the numbers will be too large to handle with the above mentioned program. To obtain more information the method of factorizing was given up in favor of using Fermat's theorem to eliminate terms which are definitely not prime numbers. We recall Fermat's little theorem:

If p is a prime number and $(a, p)=1$ then $a^{p-1} \equiv 1 \pmod{p}$.

$a^{n-1} \equiv 1 \pmod{n}$ is therefore a necessary but not sufficient condition for n to be a prime number. If n fills the congruence without being a prime number then n is called a pseudo prime to the base a, psp(a). We will proceed to find all terms in the sequence which fill the congruence

$$a^{t_n-1} \equiv 1(\mathrm{mod}\, t_n)$$

for $50 \le n \le 200$. t_{200} is a 513 digit number so we need to reduce the powers of a to the modulus t_n gradually as we go along. For this purpose we write t_n-1 to the base 2:

$$t_n-1 = \sum_{k=1}^{m} \delta(k) \cdot 2^k \text{ , where } \delta(k) \,\varepsilon\, \{0,1\}$$

From this we have

$$a^{t_n-1} = \prod_{k=1}^{m} a^{\delta(k)\cdot 2^k}$$

Table 2. Prime factorization of prime-product terms

#	P	L	N=p#+1 and its factors
1	2	1	3 Prime number
2	3	1	7 Prime number
3	5	2	31 Prime number
4	7	3	211 Prime number
5	11	4	2311 Prime number
6	13	5	30031 = 59· 509
7	17	6	510511 =19 · 97 · 277
8	19	7	9699691 =347 · 27953
9	23	9	223092871 =317 · 703763
10	29	10	6469693231 =331 · 571 · 34231
11	31	12	200560490131 Prime number
12	37	13	7420738134811 =181 · 60611 · 676421
13	41	15	304250263527211 =61 · 450451 · 11072701
14	43	17	13082761331670031= 167·78339888213593
15	47	18	614889782588491411 = 953 · 46727 · 13808181181
16	53	20	32589158477190044731 = 73 · 139 · 173 · 18564761860301
17	59	22	1922760350154212639071 =277 · 3467 · 105229 · 19026377261
18	61	24	117288381359406970983271 =223 · 525956867082542470777
19	67	25	7858321551080267055879091 =54730729297 · 143581524529603
20	71	27	557940830126698960967415391 = 1063 · 303049 · 598841 · 2892214489673
21	73	29	40729680599249024150621323471 = 2521 · 16156160491570418147806951
22	79	31	3217644767340672907899084554131 = 22093 · 1503181961 · 96888414202798247
23	83	33	267064515689275851355624017992791 = 265739 · 1004988035964897329167431269
24	89	35	23768741896345550770650537601358311 = 131 · 1039 · 2719 · 64225891884294373371806141
25	97	37	2305567963945518424753102147331756071 = 2336993 · 13848803 · 71237436024091007473549
26	101	39	232862364358497360900063316880507363071 = 960703 · 242387464553038099079594127301057
27	103	41	23984823528925228172706521638692258396211 = 2297 · 9700398839 · 179365737007 · 6001315443334531
28	107	43	2566376117594999414479597815340071648394471 = 149 ·13203797 · 3050126449106313 7 ·42767843651083711
29	109	45	279734996817854936178276161872067809674997231 = 334507 · 1290433 · 648046444234299714623177554034701
30	113	47	31610054640417607788145206291543662493274686991 = 5122427 · 2025436786007·3046707595069540247157055819
31	127	49	4014476939333036189094441199026045136645885247731 = 1543 · 49999 · 552001 · 57900988201093 ·1628080529999073967231
32	131	51	525896479052627740771371797072411912900610967452631 = 1951 · 22993 · 117232318594730141449323454664151437282666 17

Table 2. Prime factorization of prime-product terms, continued.

#	P	L	N=p#+1 and its factors
33	137	53	72047817630210000485677936198920432067383702541010311 = 881 · 1657 · 32633677 · 160823938621 · 5330099340103 · 1764291759303233
34	139	56	10014646650599190067509233131649940057366334653200433091 = 6782799590055288824986881487 · 1476476861454424513922 4580493
35	149	58	1492182350939279320058875736615841068547583863326864530411 = 87549524399 · 65018161573521013453 · 262140076844134219184937113
36	151	60	225319534991831177328890236228992001350685163362356544091911 = 23269086799180847 · 9683213481319911991636641541802024271084713
37	157	62	35375166993717494840635767087951744212057570647889977422429871 = 1381 · 1867 · 8311930927 · 3889386796857058 3 · 4244020187544088048911 3304753
38	163	64	5766152219975951659023630035336134306565384015606066319856068811 = 1361 · 2141147272105608 29 · 322670192674022105 17 · 613228865630544238382107
39	167	66	962947420735983927056946215901134429196419130606213075415963491271 = 205590139 · 53252429177 · 706457633956676 3 ·
40	173	69	166589903787325219380851695350896256250980509594874862046961683989711 = 62614127 · 2660580156093611580352333193927566158528098772260689062181793
41	179	71	29819592777931214269172453467810429868925511217482600306406141434158091 =601 · 1651781 · 8564177 · 358995947 · 1525310189119 ·6405328664096618954809029861252251
42	181	73	5397346292805549782720214077673687806275517530364350655459511599582614291 =107453 · 5634838141 ·89141572809641011233448913965712571636329746284031740286 67
43	191	76	1030893141925860008499560888835674370998623848299590975192766715520279329391 =32999 · 175603474759 · 77148541513247 ·23059614664373239595985304158624233162271520 33
44	193	78	198962376391690981640415251545285153602734402721821058212203976095413910572271 =21639496447 · 79791259059673394950188 77·115230777162797580440201621017774536159 09
45	197	80	3919558814916312338316180455442117525973867733619874846780418329079654038273719 1=521831·50257723 · 1601684368321 ·390811702432625410 27 ·23875913958369977158572653160969521
46	199	82	7799922041683461553249199106329813876687996789903550945093032474868511536164700811 =467 ·10723 · 57622771 · 5876645549 ·9458145520867 ·48632595443062609609719222040521494786550384 7
47	211	85	1645783550795210387735581011435590727981167322669649249414629852197255934130751870911 =1051 · 2179 · 16333 · 43283699 · 75311908487 ·29281271068483 9 ·46096596672866469293430334044872907384889
48	223	87	3670097318273319164650345655501367323398003129553317826194624570399880733111576672129 31=13867889468159 ·2646471423571660867679159849289670356488810003605334293061946803757288050 9
49	227	89	83311209124804345037562846379881038241134671040860314654617977748077292641632790457335111 = 3187 · 31223 · 1737142793 · 11463039340315601 ·97310450547044696930911 3 · 43206785807567189232875099500379

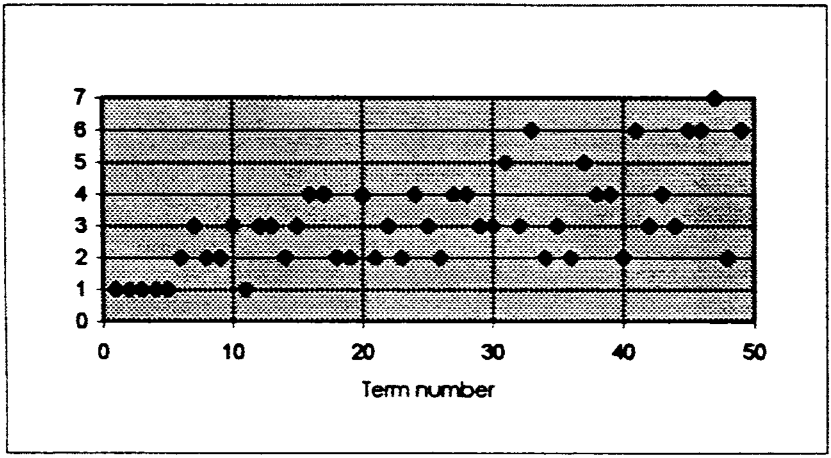


Diagram 3. The number of prime factors in the first 49 terms of the prime-product sequence.

This product expression for a^{t_n-1} is used in the following *Ubasic* program to carry out the reduction of a^{t_n-1} modulus t_n. Terms for which $\delta(k)=0$ are ignored in the expansion were the exponents k are contained in the array E%(). The residue modulus t_n is stored in F. In the program below the reduction is done to base A=7.

```
100 dim E%(1000)
110 M=N-1:I%=0
120 T=1:J%=0
130 while (M-T)>=0
140 inc J%:T=2*T
150 wend
160 dec J%:M=M-T\2:inc I%:E%(I%)=J%
170 if M>0 then goto 120
180 F=1
190 for J%=1 to I%
200 A=7
210 for K%=1 to E%(J%)
240 A=(A^2)@N
250 next
260 F=F*A:F=F@N
270 next
```

This program revealed that there are at most three terms t_n of the sequence in the interval $50 \le n \le 200$ which could be prime numbers. These are:

<u>Term #75. N=379#+1. N is a 154 digit number.</u>
N=171962010545840643348334056831754301958457563589574256043877110505832165523856261308397965147955578800999455782202456522693290629520826275682227 5663694111

<u>Term #171. N=1019#+1. N is a 425 digit number.</u>
N=2040406899301637419454246417277460769565979711742312191322713103233902616917592990224445375741046872884292986227160556781882168549067666198538983995862280246598688137613940413837615309610314083466556364674016027975521231750135686300363861239066166840623542231178374239051052658725702650030269683479324852673430580163416594870250636717670123329806461666355371697542904875157559715041738106393425568912448602949290896664474 7931

<u>Term #172. N=1021#+1. N is a 428 digit number.</u>
N=2083255444186971805262785592040287445726865285688900747340490078401814571872862443019158728631608857214863138937930928474301694088598087188708302659775388131777260588503833162528205231112130679219354048332170364563007177616888535712671502325086556344276636618033120098071124764558942405680905346832390674579572622346848343362525900088741195919732397361348834503191305877535868469057614606627687505859610023611226005494428 7636531

The last two primes or pseudo primes are remarkable in that they are generated by the prime twins 1019 and 1021.

Summary of results: The number of primes q among the first 200 terms of the prime-product sequence is given by $6 \le q \le 9$. The six confirmed primes are terms numero 1, 2, 3, 4, 5 and 11. The three terms which are either primes or pseudo primes are terms numero 75, 171 and 172. The latter two are the terms 1019#+1 and 1021#+1.

II.3 The Square-Product Sequence

Definition: The terms of the square-product sequence are defined through $\{t_n : t_n = (n!)^2+1\}$

This sequence has a structure which is similar to the prime-product sequence. The analysis is therefore carried out almost identically to the one done for the prime-product sequence. We merely have to state the results and compare them.

The sequence begins {2, 5, 37, 577, 14401, 518401, ... }.As for the prime-product sequence the question of how many are prime numbers has been raised and we may never know. There are similarities between these two sequences. There are quite a few primes among the first terms. After that they become more and more rare. Complete factorization of the 37 first terms of the square-factorial sequence was obtained and has been used in diagram 4 which should be compared with the corresponding diagram 3 for the prime-product sequence.

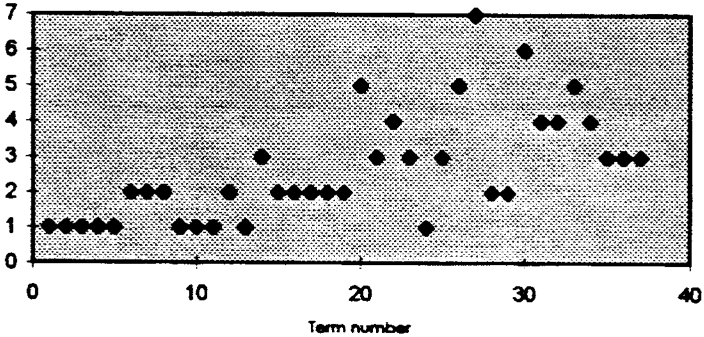


Diagram 4. The number of prime factors in the first 40 terms of the square-factorial sequence.

Diagram 4 is based on table 3 which shows the prime factorization of the 40 first terms in the square-factorial sequence. The number of factors of each term is denoted f. The factorization is not complete for terms numero 38 and 39. A +-sign in the column for f in table 3 indicates that the last factor is not a prime. The terms of this sequence are in general much more time consuming to factorize than those of the prime-product sequence which accounts for the more limited results in this section. Using the same method as for the prime-

Table 3. Prime factorization of square-factorial terms.

n	L	f	N=(n!)²+1 and its factors
1	1	1	2
2	1	1	5
3	2	1	37
4	3	1	577
5	5	1	14401
6	6	2	518401=13·39877
7	8	2	25401601= 101· 251501
8	10	2	1625702401= 17· 95629553
9	12	1	131681894401
10	14	1	13168189440001
11	16	1	1593350922240001
12	18	2	229442532802560001=101· 2271708245569901
13	20	1	38775788043632640001
14	22	3	7600054456551997440001= 29· 109· 2404319663572286441
15	25	2	1710012252724199424000001=1344169·1272170577304043929
16	27	2	437763136697395052544000001=149· 2938007628841577533852349
17	30	2	126513546505547170185216000001= 9049·13980942259426143240713449
18	32	2	40990389067797283140009984000001= 37·1107848353183710355135404972973
19	35	2	14797530453474819213543604224000001=710341·20831587158104092560535861261
20	37	5	5919012181389927685417441689600000001=41·10657· 86816017· 348046955609·448324749841
21	40	3	2610284371992958109269091785113600000001= 61·157·2725576247251705240961774861766631513
22	43	4	1263377636044591724886240423994982400000001= 337· 8017· 514049836440277481· 909674823323537849
23	45	3	668326769467589022464821184293345689600000001= 509·15448374629· 84994002604532747687401741723441
24	48	1	384956219213331276939737002152967117209600000001
25	51	3	240597637008332048087335626345604448256000000000001= 941· 815769831908479758733· 313425331349331290243399417
26	54	5	162644002617632464507038883409628607021056000000000001= 53· 53· 418633· 6017159668589· 229858897128760962225564623301797
27	57	7	118567477908254066625631346005619254518349824000000000001= 113· 42461· 745837· 2460281· 7566641· 15238649· 116793504008451126962009
28	59	2	92956902680071188234494975268405495542386262016000000000001= 212259034657663450 9· 43794085292997939303952241474982753464389
29	62	2	7817675515393986930521027420072902175114684635545600000000000001= 1717078604732075883498 37· 455289320701414063716469396531758248773
30	65	6	70359079638545882374689246780656119576032161719910400000000000001= 61· 1733· 15661· 359525849· 10063638112656869011006 9· 117459224951820775953789 7
31	68	4	676150755326425929620763661562105309125669074128338944000000000000001= 353· 422041· 13400767181· 3386760818094840908530582079383219157032466782167 7
32	71	4	6923783734542601519316619894395958365446851319074190786560000000000000001= 10591621681· 6415450838021· 5223032939146600012049 69·19508823885853555320254 29
33	74	5	7540000486916893054535799064997198659971621086471793766563840000000000000001= 37· 3121· 4421· 40733328828459362 53· 36258135123244480427387450762108578223148 052301

Table 3 continued.

n	L	f	$N=(n!)^2+1$ and its factors
34	77	4	8716240562875928371043383719136761650927193975961393594147799040000000000000 01=193· 3217· 866100731693·39452143443145645231476894644096901291197410624286 816576197
35	81	3	1067739468952301225452814505594253302238581262055270715283105382400000000000 000001=317·373·903019653886808488978285455632355360863474820117616321989077 7 16189815715361
36	84	3	1383790351762182388186847599250152279701201315623630847006904575590400000000 00000001=73· 57986941373 ·32690174316982778045052076286249517817882480631366 14675401163773414986 9
37	87	3	1894408991562427689427794363373458470910944601088750629552452363983257600000 00000000001=127406364297881· 4910557119433812802191010 9· 30279720114524038 42 829243076981427205232764306 9
38	90	4+	2735526583816145583533735060711274031995404003972155909073741213591823974 40 000000000000001=233·757·155091908074914281217009488590680063725424167227317 9 03236388341918450625316785821602 1
39	93	4+	4160735933984357432554811027341847802665009490041649137701160385873164265 06 24000000000000000001=61·100454575774 1·67900128269327066547084588212509348973 29 615368972937285510229243157273033617801
40	96	4	6657177494374971892087697643746956484264015184066638620321856617397062824 09 984000000000000000001=89· 701· 187100101949·5703061928798691519563156731422 22 3621393496539544379492647728174834902025511344 1

product sequence the terms t_n in the interval $40<n\leq200$ which may possible be primes were identified. There are only two of them:

<u>Term #65. $N=(65!)^2+1$. N is a 182 digit number.</u>
6802374028907832895045078197262220379290257695327135803427938010402710065
2464382649659623724446578151412858996571534385340563792951822384455180747
800576000000000000000000000000000001

<u>Term #76 $N=(76!)^2+1$. N is a 223 digit number.</u>
3555090270010747854202513135770772648194325666925541647977005250280050084
1772266884421391665890651643920912930369944999452531006264950776782697850
7198658011625298409931764786386381150617600000000000000000000000000000000
0001

II.4 The Smarandache Prime-Digital Sub-Sequence

Definition: The prime-digital sub-sequence is the set :
$\{M=a_0+a_1\cdot10+a_2\cdot10^2+\ldots a_k10^k$:M is prime, all digits $a_0, a_1, a_2\ldots a_k$ are primes$\}$

The first terms of this sequence are {2, 3, 5, 7, 23, 37, 53, 73, ... }. Sylvester Smith [1] conjectured that this sequence is infinite. In this paper we will prove that this sequence is in fact infinite. Let's first calculate some more terms of the sequence and at the same time find how many terms there is in the sequence in a given interval, say between 10^k and 10^{k+1} .The program below is written in *Ubasic.* One version of the program has been used to produce table 4 showing the first 100 terms of the sequence. The output of the actual version has been used to produce the calculated part of table 5 which we are going to compare with the theoretically estimated part in the same table.

Ubasic program

```
10  point 2
20  dim A%(6),B%(4)
30  for I%=1 to 6:read A%(I%):next
40  data 1,4,6,8,9,0                          'Digits not allowed stored in A%()
50  for I%=1 to 4:read B%(I%):next
60  data 2,3,5,7                              'Digits allowed stored in B%()
70  for K%=1 to 7                             'Calc. for 7 separate intervals
80  M%=0:N=0
90  for E%=1 to 4                             'Only 2,3,5,7 allowed as first digit
100 P=B%(E%)*10^K%:P0=P:S=(B%(E%)+1)*10^K%:gosub 150
110 next
120 print K%,M%,N,M%/N
130 next
140 end
150 while P<S
160 P=nxtprm(P):P$=str(P)                     'Select prime and convert to string
170 inc N                                     'Count number of primes
180 L%=len(P$):C%=0                           'C% will be set to 1 if P not member
190 for I%=2 to L%
200 for J%=1 to 6                             'This loop examines each digit of P
210 if val(mid(P$,I%,1))=A%(J%) then C%=1
220 next:next
230 if C%=0 then  inc M%                      'If criteria filled count member (m%)
240 wend
250 return
```

Table 4. The first 100 terms in the prime-digital sub sequence.

2	3	5	7	23	37	53	73	223	227
233	257	277	337	353	373	523	557	577	727
733	757	773	2237	2273	2333	2357	2377	2557	2753
2777	3253	3257	3323	3373	3527	3533	3557	3727	3733
5227	5233	5237	5273	5323	5333	5527	5557	5573	5737
7237	7253	7333	7523	7537	7573	7577	7723	7727	7753
7757	22273	22277	22573	22727	22777	23227	23327	23333	23357
23537	23557	23753	23773	25237	25253	25357	25373	25523	25537
25577	25733	27253	27277	27337	27527	27733	27737	27773	32233
32237	32257	32323	32327	32353	32377	32533	32537	32573	33223

Table 5. Comparison of results.

k	1	2	3	4	5	6	7
Computer count:							
m	4	15	38	128	389	1325	4643
log(m)	0.6021	1.1761	1.5798	2.1072	2.5899	3.1222	3.6668
n	13	64	472	3771	30848	261682	2275350
m/n	0.30769	0.23438	0.08051	0.03394	0.01261	0.00506	0.00204
Theoretical estimates:							
m	4	11	34	109	364	1253	4395
log(m)	0.5922	1.0430	1.5278	2.0365	2.5615	3.0980	3.6430
n	7	55	421	3399	28464	244745	2146190
m/n	0.50000	0.20000	0.08000	0.03200	0.01280	0.00512	0.00205

Theorem:
The Smarandache prime-digital sub sequence is infinite.

Proof:
We recall the prime counting function $\pi(x)$. The number of primes $p \leq x$ is denoted $\pi(x)$. For sufficiently large values of x the order of magnitude of

$\pi(x)$ is given by $\pi(x) \approx \dfrac{x}{\log x}$. Let a and b be digits such that $a>b\neq 0$ and $n(a,b,k)$ the approximate number of primes in the interval $(b\cdot 10^k, a\cdot 10^k)$. Applying the prime number counting theorem we then have:

$$n(a,b,k) \approx \frac{10^k}{k}\left(\frac{a}{\log 10 + \frac{\log a}{k}} - \frac{b}{\log 10 + \frac{\log b}{k}}\right) \tag{1}$$

Potential candidates for members of the prime-digital sub sequence will have first digits 2,3,5 or 7, i.e. for a given k they will be found in the intervals $(2\cdot 10^k, 4\cdot 10^k)$, $(5\cdot 10^k, 6\cdot 10^k)$ and $(7\cdot 10^k, 8\cdot 10^k)$. The approximate number of primes $n(k)$ in the interval $(10^k, 10^{k+1})$ which might be members of the sequence is therefore:

$$n(k)=n(4,2,k)+n(6,5,k)+n(8,7,k) \tag{2}$$

The theoretical estimates of n in table 5 are calculated using (2) ignoring the fact that results may not be all that good for small values of k.

We will now find an estimate for the number of candidates $m(k)$ which qualify as members of the sequence. The final digit of a prime number >5 can only be 1,3,7 or 9. Assuming that these will occur with equal probability only half of the candidates will qualify. The first digit is already fixed by our selection of intervals. For the remaining k-1 digits we have ten possibilities, namely 0,1,2,3,4,5,6,7,8 and 9 of which only 2,3,5 and 7 are good. The probability that all k-1 digits are good is therefore $(4/10)^{k-1}$. The probability q that a candidate qualifies as a member of the sequence is

$$q = \frac{1}{2}\cdot\left(\frac{4}{10}\right)^{k-1} \qquad 3)$$

The estimated number of members of the sequence in the interval $(10^k, 10^{k+1})$ is therefore given by $m(k)=q\cdot n(k)$. The estimated values are given in table 5. A comparison between the computer count and the theoretically estimated

values shows a very close fit as can be seen from diagram 5 where $\log_{10}$ m is plotted against k.

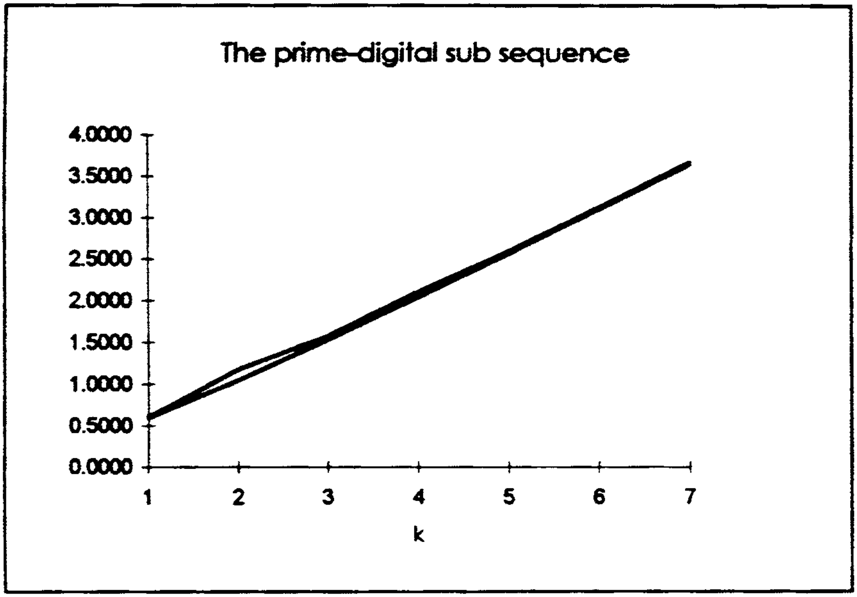


Diagram 5. $\log_{10}$ m as a function of k. The upper curve corresponds to the computer count.

For large values of k we can ignore the terms $\frac{\log a}{k}$ and $\frac{\log b}{k}$ in comparison with log 10 in (1). For large k we therefore have

$$n(a,b,k) \approx \frac{(a-b)10^k}{k\log 10} \tag{1'}$$

and (2) becomes

$$n(k) \approx \frac{4 \cdot 10^k}{k \log 10} \tag{2'}$$

Combining this with (3) we get

$$m(k) \approx \frac{5 \cdot 2^{2k}}{k \log 10} \tag{4}$$

From which we see (apply for instance l'Hospital's rule) that $m(k) \to \infty$ as $k \to \infty$. A fortiori the prime-digital sub sequence is infinite.

Chapter III

Non-Recursive Sequences

III.1 Smarandache Primitive Numbers

Definition: For a given prime p and positive integer n the Smarandache primitive number $S_p(n)$ is the smallest positive integer such that $S_p(n)!$ is divisible by p^n.

This sequence is important for the calculation of the Smarandache function S(n) which is defined as the smallest integer such that S(n)! is divisible by n. We note that $S_p(n)=S(p^n)$.

It follows immediately from the definition that the sequence $S_p(n)$ for a given prime p consists of multiples of p. Furthermore the definition implies that the sequence is non-descending, i.e. $S_p(n)\leq S_p(n+1)$.

It is evident from the definition that

$$S_p(n)=np \text{ for } n\leq p \tag{1}$$

Upper and lower bounds for $S_p(n)$ have been established by Pål Grönås [1]:

$$(p-1)n+1\leq S_p(n)\leq pn \tag{2}$$

Let's asssume $S_p(n-1)<S_p(n)$ and $p^k | S_p(n)$ then it follows that $S_p(n+k-1)= S_p(n+k-2)= \ldots = S_p(n+1)= S_p(n)$. This observation is expressed in the following way in *Some Notions and Questions in Number Theory [2]* " Curious property: this is the sequence of multiples of p, each number being repeated as many times as its exponent (of power p) is."

Algorithms for the calculation of $S_p(n)$ are important as a stepping stone for the calculation of the Smarandache function. The lack of a closed formula makes this a very interesting topic. Three methods will be described, implemented and compared.

Method 1.

Let's denote $S_p(n)=m$. We want to calculate the smallest value of m for which $1\cdot 2\cdot 3\cdot \ldots \cdot m \equiv 0 \pmod{p^n}$. This is carried out in the following Ubasic program:

```
10 input "p,n";P,N
20 if N<=P then M=N*P:goto 80
30 Y=1:Z=P^N
40 while Y>0
50 inc M
60 Y=Y*M@Z          'The factorial is reduced mod z each time
70 wend             'the loop is executed.
80 print P,N,M
90 end
```

This program is effective for small values of n. As we move on to large exponents the increasingly large modulus slows down the program. A study of the compexity of this short and elegant program, which can equally well be used to calculate S(n), has been carried out by S.and T. Tabirca [3].

Method 2.

When $n \le p$ we have $S_p(n)=np$. For $n>p$ our potential solutions are multiples of p. The first multiple we need to examine is p^2. For a given multiple m of p we determine first the largest power of p which is less than m, i.e. we determine k so that $p^k \le m < p^{k+1}$. We then continue by counting the number of factors p in m!. This is given by $S = [m/p] + [m/p^2] + [m/p^3] + \ldots + [m/p^k]$. If $S<n$ then we proceed to the next multiple of p and so on until we arrive at $S=0$ in which case $m=S_p(n)$.

The following *UBASIC* program has been used to tabulate $S_p(n)$ as well as the Smarandache function in the *Smarandache Function Journal* [4].

```
10  input "p,n";P,N
20  if N<=P then M=N*P:goto 130    'Sp(n)=np in this case
30  I=P-1                           'i =p-1 so that m starts at p²
40  while S<N                       'in line 60
50  inc I
60  M=I*P                     'The starting point for m=p²
70  S=0
80  K=floor(log(M)/log(P))    'Determine the largest k for which
90  for J=1 to K              'pᵏ<m
100  S=S+floor(M/P^J)         'Count and add the number times
110  next                     'pʲ occurs in m!
120  wend
130  print P,N,M
140  end
```

Method 3.

This method is based on a theorem due to C. Dumitrescu and V.Seleacu [5]. They prove the following:

<u>Theorem.</u> If for a given prime p the integer n is expressed in the number base $b_1=1$, $b_2=1+p$, $b_3=1+p+p^2$, ..., $b_k=1+p+ \ldots p^{k-1}$, resulting in $n=c_1b_1+ c_2b_2+ \ldots + c_kb_k$, where the first non-zero coefficient $c_i \leq p$ and $c_j<p$ for $i<j\leq k$, then $S_p(n)=c_1p + c_2p^2 + \ldots c_kp^k$.

In the program below the number base is generated through the recursive formula $b_k=p\cdot b_{k-1}+1$.

```
10  dim C(100),B(100)
20  input "p,n";P,N
30  if N<=P then M=P*N:goto 150
40  B(1)=1:I=1
50  while N>=B(I)
60  inc I
70  B(I)=B(I-1)*P+1                'Calculate b2, b3, etc
80  wend
90  K=I-1
100  for I=K to 1 step -1
```

```
110  C(I)=N\B(I)          'Calculate the coefficients
120  N=res                'c1, c2, ...
130  M=M+C(I)*P^i         'Express Sp(n) using these
140  next                 'coefficients
150  print M
160  end
```

Results.

Which of these three programs is the most effective? Since $S_p(n)=np$ when $n \leq p$ all three programs will be equally efficient for $n \leq p$. The programs were therefore tested by calculating $S_p(n)$ for $n=p+1$, $p+2$, ... $p+50$ for $p=2,3,5,7,\ldots 229$ (the first 50 primes).The largest number manipulated in the programs is therefore 229^{279}, which is a 559-digit number. The programs were fitted as routines in the same in- and output program. Short integers were used whenever possible. To calculate these 2500 values of $S_p(n)$ method I took 31 m 26 s, method II 22 s and method III 11 s on a Pentium 100 Mhz laptop. The time t was measured in milliseconds. In figure 1 ln t is plotted against the number of primes for which the programs have been executed. It is seen that method III is the most effective. It runs about twice as fast as method II. Method I is effective for small primes but slows down considerably when the modulus p^n increases.

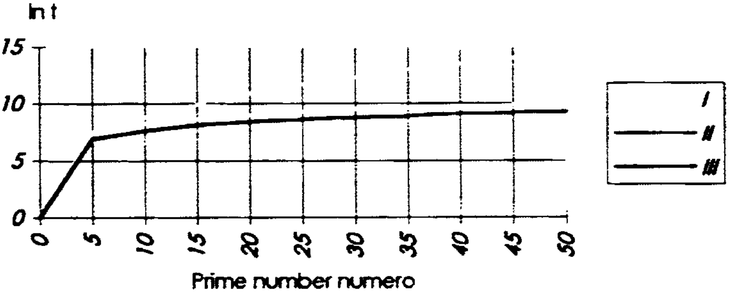


Figure 1. Comparison of execution times for methods I to III.

Table 1. $S_p(n)$ for n=1, 2, ... 48 and p≤47

n	p														
6	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
2	4	6	10	14	22	26	34	38	46	58	62	74	82	86	94
3	4	9	15	21	33	39	51	57	69	87	93	111	123	129	141
4	6	9	20	28	44	52	68	76	92	116	124	148	164	172	188
5	8	12	25	35	55	65	85	95	115	145	155	185	205	215	235
6	8	15	25	42	66	78	102	114	138	174	186	222	246	258	282
7	8	18	30	49	77	91	119	133	161	203	217	259	287	301	329
8	10	18	35	49	88	104	136	152	184	232	248	296	328	344	376
9	12	21	40	56	99	117	153	171	207	261	279	333	369	387	423
10	12	24	45	63	110	130	170	190	230	290	310	370	410	430	470
11	14	27	50	70	121	143	187	209	253	319	341	407	451	473	517
12	16	27	50	77	121	156	204	228	276	348	372	444	492	516	564
13	16	27	55	84	132	169	221	247	299	377	403	481	533	559	611
14	16	30	60	91	143	169	238	266	322	406	434	518	574	602	658
15	16	33	65	98	154	182	255	285	345	435	465	555	615	645	705
16	18	36	70	98	165	195	272	304	368	464	496	592	656	688	752
17	20	36	75	105	176	208	289	323	391	493	527	629	697	731	799
18	20	39	75	112	187	221	289	342	414	522	558	666	738	774	846
19	22	42	80	119	198	234	306	361	437	551	589	703	779	817	893
20	24	45	85	126	209	247	323	361	460	580	620	740	820	860	940
21	24	45	90	133	220	260	340	380	483	609	651	777	861	903	987
22	24	48	95	140	231	273	357	399	506	638	682	814	902	946	1034
23	26	51	100	147	242	286	374	418	529	667	713	851	943	989	1081
24	28	54	100	147	242	299	391	437	529	696	744	888	984	1032	1128
25	28	54	105	154	253	312	408	456	552	725	775	925	1025	1075	1175
26	30	54	110	161	264	325	425	475	575	754	806	962	1066	1118	1222
27	32	57	115	168	275	338	442	494	598	783	837	999	1107	1161	1269
28	32	60	120	175	286	338	459	513	621	812	868	1036	1148	1204	1316
29	32	63	125	182	297	351	476	532	644	841	899	1073	1189	1247	1363
30	32	63	125	189	308	364	493	551	667	841	930	1110	1230	1290	1410
31	32	66	125	196	319	377	510	570	690	870	961	1147	1271	1333	1457
32	34	69	130	196	330	390	527	589	713	899	961	1184	1312	1376	1504
33	36	72	135	203	341	403	544	608	736	928	992	1221	1353	1419	1551
34	36	72	140	210	352	416	561	627	759	957	1023	1258	1394	1462	1598
35	38	75	145	217	363	429	578	646	782	986	1054	1295	1435	1505	1645
36	40	78	150	224	363	442	578	665	805	1015	1085	1332	1476	1548	1692
37	40	81	150	231	374	455	595	684	828	1044	1116	1369	1517	1591	1739
38	40	81	155	238	385	468	612	703	851	1073	1147	1369	1558	1634	1786
39	42	81	160	245	396	481	629	722	874	1102	1178	1406	1599	1677	1833
40	44	81	165	245	407	494	646	722	897	1131	1209	1443	1640	1720	1880
41	44	84	170	252	418	507	663	741	920	1160	1240	1480	1681	1763	1927
42	46	87	175	259	429	507	680	760	943	1189	1271	1517	1681	1806	1974
43	48	90	175	266	440	520	697	779	966	1218	1302	1554	1722	1849	2021
44	48	90	180	273	451	533	714	798	989	1247	1333	1591	1763	1849	2068
45	48	93	185	280	462	546	731	817	1012	1276	1364	1628	1804	1892	2115
46	48	96	190	287	473	559	748	836	1035	1305	1395	1665	1845	1935	2162
47	50	99	195	294	484	572	765	855	1058	1334	1426	1702	1886	1978	2209
48	52	99	200	294	484	585	782	874	1058	1363	1457	1739	1927	2021	2209

In table 1 consecutive squares have been emphasized by frames. The series in light frames follow the rule $S_p(p)= S_p(p+1)=p^2$ which is obvious from the definition. The series in bold frames is less obvious and this study will be concluded by proving the following:

$$S_p(4p+4)= S_p(4p+3)=4p^2 \text{ or } S(p^{4(p+1)})=S(p^{4p+3})=(2p)^2$$

Proof: Applying the theorem of Dumitrescu and Seleacu we have:

$4p+4=4\cdot(1+p)$, i.e. $c_1=0$ and $c_2=4$ from which

$S_p(4p+p)=0\cdot p+4\cdot p^2=(2p)^2$

and

$4p+3=p\cdot 1+3\cdot(1+p)$, i.e. $c_1=p$ and $c_2=3$ from which

$S_p(4p+3)=p\cdot p+3\cdot p^2=(2p)^2$

III.2 The Smarandache Function S(n)

Definition: S(n) is the smallest integer such that S(n)! is divisible by n.

The properties of this function have been subject to detailed studies in many papers. Some of these properties are listed here. When needed n is represented in the form $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

1. $S(n)\leq n$

2. p is prime iff $S(p)=p$

3. If $m\cdot n\neq 0$, then $S(mn)\leq S(n)\cdot S(m)$

4. If $(m,n)=1$, then $S(mn)=\max\{S(m),S(n)\}$
5. $S(n)=\max\{S_{p_1}(\alpha_1),\ S_{p_2}(\alpha_2),\ \ldots S_{p_k}(\alpha_k)\}$, where $S_p(n)$ are Smarandache primitive numbers as defined in section II.1

6. If n is square free, i.e. $\alpha_1=\alpha_2=\ldots=\alpha_k$, then $S(n)=\max\{p_1, p_2, \ldots p_k\}$

The purpose of this section, however, is to give an effective computer algorithm[1] for the calculation of S(n) and to provide a table for the first 1000 values of this function.

```
10  'A UBASIC program for online calculation of S(n)
20  dim F(10),E%(20)
30  input "N ";N
40  gosub 70
50  print "S(n)= ";S
60  end
70  N1=N:gosub 170
```

N is factorized and the prime factors stored in F() with the corresponding exponents in E%()

```
80  if F(1)=N then S=N:goto 160
```

If F(1)=1 then n is prime and consequently S(n)=n

```
90  for J%=1 to K%
100  if E%(J%)>1 then P2=F(J%):N2%=E%(J%):gosub 260:F(J%)=M
110  next
```

For prime factors of n with exponents larger than 1 F(J%)=P2 is replaced by F(j%)=M which is calculated by one of the routines described earlier (here method 2)

```
120  S=0
```

[1] The author was inspired to do this after several requests to help provide values for this function for large values of n.

```
130 for J%=1 to K%
140 if F(J%)>S then S=F(J%)
150 next
```

The largest value of F() is the value of S(n)

```
160 return
170 'Factorisation of N1, output f(i)^e%(i)
```

A simple routine to factorize N1

```
180 K%=0:for I%=1 to 10:F(I%)=0:next
190 for I%=1 to 20:E%(I%)=0:next
200 while N1>1
210 P=prmdiv(N1)
220 if P=F(K%) then inc E%(K%) else inc K%:F(K%)=P:inc E%(K%)
230 N1=N1\P
240 wend
250 return
260 'Calculation of Sp(n). In: p2,n2%. Out: m. This routine is
    documented in section II.1
270 I2=0:S2%=0
280 while S2%<N2%
290 inc I2
300 M=I2*P2
310 S2%=0
320 K2=floor(log(M)/log(P2))
330 for J2=1 to K2
340 S2%=S2%+floor(M/P2^J2)
350 next
360 wend
370 return
```

This is the bare minimum program. When used in a special problem the number of instructions tend to increase considerably because of "safety valves" and data flow from and to other processes.

Table 2. The Smarandache function

n	0	1	2	3	4	5	6	7	8	9
		0	2	3	4	5	3	7	4	6
1_	5	11	4	13	7	5	6	17	6	19
2_	5	7	11	23	4	10	13	9	7	29
3_	5	31	8	11	17	7	6	37	19	13
4_	5	41	7	43	11	6	23	47	6	14
5_	10	17	13	53	9	11	7	19	29	59
6_	5	61	31	7	8	13	11	67	17	23
7_	7	71	6	73	37	10	19	11	13	79
8_	6	9	41	83	7	17	43	29	11	89
9_	6	13	23	31	47	19	8	97	14	11
10_	10	101	17	103	13	7	53	107	9	109
11_	11	37	7	113	19	23	29	13	59	17
12_	5	22	61	41	31	15	7	127	8	43
13_	13	131	11	19	67	9	17	137	23	139
14_	7	47	71	13	6	29	73	14	37	149
15_	10	151	19	17	11	31	13	157	79	53
16_	8	23	9	163	41	11	83	167	7	26
17_	17	19	43	173	29	10	11	59	89	179
18_	6	181	13	61	23	37	31	17	47	9
19_	19	191	8	193	97	13	14	197	11	199
20_	10	67	101	29	17	41	103	23	13	19
21_	7	211	53	71	107	43	9	31	109	73
22_	11	17	37	223	8	10	113	227	19	229
23_	23	11	29	233	13	47	59	79	17	239
24_	6	241	22	12	61	14	41	19	31	83
25_	15	251	7	23	127	17	10	257	43	37
26_	13	29	131	263	11	53	19	89	67	269
27_	9	271	17	13	137	11	23	277	139	31
28_	7	281	47	283	71	19	13	41	8	34
29_	29	97	73	293	14	59	37	11	149	23
30_	10	43	151	101	19	61	17	307	11	103
31_	31	311	13	313	157	7	79	317	53	29
32_	8	107	23	19	9	13	163	109	41	47
33_	11	331	83	37	167	67	7	337	26	113
34_	17	31	19	21	43	23	173	347	29	349
35_	10	13	11	353	59	71	89	17	179	359
36_	6	38	181	22	13	73	61	367	23	41
37_	37	53	31	373	17	15	47	29	9	379
38_	19	127	191	383	8	11	193	43	97	389
39_	13	23	14	131	197	79	11	397	199	19
40_	10	401	67	31	101	9	29	37	17	409
41_	41	137	103	59	23	83	13	139	19	419
42_	7	421	211	47	53	17	71	61	107	13
43_	43	431	9	433	31	29	109	23	73	439
44_	11	14	17	443	37	89	223	149	8	449
45_	10	41	113	151	227	13	19	457	229	17
46_	23	461	11	463	29	31	233	467	13	67
47_	47	157	59	43	79	19	17	53	239	479
48_	8	37	241	23	22	97	12	487	61	163
49_	14	491	41	29	19	11	31	71	83	499

50_	15	167	251	503	7	101	23	26	127	509

Table 2 ctd. The Smarandache function

n	0	1	2	3	4	5	6	7	8	9
51_	17	73	12	19	257	103	43	47	37	173
52_	13	521	29	523	131	10	263	31	11	46
53_	53	59	19	41	89	107	67	179	269	14
54_	9	541	271	181	17	109	13	547	137	61
55_	11	29	23	79	277	37	139	557	31	43
56_	7	17	281	563	47	113	283	9	71	569
57_	19	571	13	191	41	23	8	577	34	193
58_	29	83	97	53	73	13	293	587	14	31
59_	59	197	37	593	11	17	149	199	23	599
60_	10	601	43	67	151	22	101	607	19	29
61_	61	47	17	613	307	41	11	617	103	619
62_	31	23	311	89	13	20	313	19	157	37
63_	7	631	79	211	317	127	53	14	29	71
64_	8	641	107	643	23	43	19	647	9	59
65_	13	31	163	653	109	131	41	73	47	659
66_	11	661	331	17	83	19	37	29	167	223
67_	67	61	8	673	337	10	26	677	113	97
68_	17	227	31	683	19	137	21	229	43	53
69_	23	691	173	11	347	139	29	41	349	233
70_	10	701	13	37	11	47	353	101	59	709
71_	71	79	89	31	17	13	179	239	359	719
72_	6	103	38	241	181	29	22	727	13	15
73_	73	43	61	733	367	14	23	67	41	739
74_	37	19	53	743	31	149	373	83	17	107
75_	15	751	47	251	29	151	9	757	379	23
76_	19	761	127	109	191	17	383	59	10	769
77_	11	257	193	773	43	31	97	37	389	41
78_	13	71	23	29	14	157	131	787	197	263
79_	79	113	11	61	397	53	199	797	19	47
80_	10	89	401	73	67	23	31	269	101	809
81_	9	811	29	271	37	163	17	43	409	13
82_	41	821	137	823	103	11	59	827	23	829
83_	83	277	13	17	139	167	19	31	419	839
84_	7	58	421	281	211	26	47	22	53	283
85_	17	37	71	853	61	19	107	857	13	859
86_	43	41	431	863	9	173	433	34	31	79
87_	29	67	109	97	23	15	73	877	439	293
88_	11	881	14	883	17	59	443	887	37	127
89_	89	11	223	47	149	179	8	23	449	31
90_	10	53	41	43	113	181	151	907	227	101
91_	13	911	19	83	457	61	229	131	17	919
92_	23	307	461	71	11	37	463	103	29	929
93	31	19	233	311	467	17	13	937	67	313
94_	47	941	157	41	59	9	43	947	79	73
95_	19	317	17	953	53	191	239	29	479	137
96_	8	62	37	107	241	193	23	967	22	19
97_	97	971	12	139	487	13	61	977	163	89
98_	14	109	491	983	41	197	29	47	19	43

99	11	991	31	331	71	199	83	997	499	37

III.3 Smarandache m-Power Residues

Definition: The m-power residue of n is the largest m-power free number which divides n. Notation $M_r(n)$.

Let's express n in the form $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots p_k^{\alpha_k}$ then $M_r(n) = p_1^{\alpha_1'} \cdot p_2^{\alpha_2'} \cdot \ldots p_k^{\alpha_k'}$, where $\alpha_i' = \min\{\alpha_i, m-1\}$ for $i=1, 2, \ldots k$.

It follows directly from the definition that $M_r(n)$ is a multiplicative arithmetic function, which is expressed in (1) below.

If $(n,m)=1$ then $M_r(nm)=M_r(n)\cdot M_r(m)$

We will only make calculations for two special cases: Cubical residues and Square residues.

III.3a Cubical Residues.

In the prime factor representation of n $\alpha_i \geq 3$ will be replaced by $\alpha_i=2$. A simple *Ubasic* routine results in table 3.

Table 3. The first 100 terms of the cubical residues sequence.

n	0	1	2	3	4	5	6	7	8	9
		1	2	3	4	5	6	7	4	9
1_	10	11	12	13	14	15	4	17	18	19
2_	20	21	22	23	12	25	26	9	28	29
3_	30	31	4	33	34	35	36	37	38	39
4_	20	41	42	43	44	45	46	47	12	49
5_	50	51	52	53	18	55	28	57	58	59
6_	60	61	62	63	4	65	66	67	68	69
7_	70	71	36	73	74	75	76	77	78	79
8_	20	9	82	83	84	85	86	87	44	89
9_	90	91	92	93	94	95	12	97	98	99

III.3b Square Residues

For n in the form $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots p_k^{\alpha_k}$ the square residue $S_r(n)$ is given by $S_r(n)=p_1 \cdot p_2 \cdot \ldots p_k$.
Apart from the multiplictive property we add two more properties of this sequence.

If $\alpha>0$ and p any prime number then $S_r(p^\alpha)=p$.

$S_r(nm)=S_r(n) \cdot S_r(m) /(S_r(n), S_r(m))$.

The first 100 terms of this sequence are given in table 4.

Table 4. The first 100 terms of the square residues sequence

n	0	1	2	3	4	5	6	7	8	9
		1	2	3	2	5	6	7	2	3
1_	10	11	6	13	14	15	2	17	6	19
2_	10	21	22	23	6	5	26	3	14	29
3_	30	31	2	33	34	35	6	37	38	39
4_	10	41	42	43	22	15	46	47	6	7
5_	10	51	26	53	6	55	14	57	58	59
6_	30	61	62	21	2	65	66	67	34	69
7_	70	71	6	73	74	15	38	77	78	79
8_	10	3	82	83	42	85	86	87	22	89
9_	30	91	46	93	94	95	6	97	14	33

In chapter VI we will return to the Smarandache function and the square residues.

References:

[1] Pål Grønås: A note on S(p^r), *Smarandache Function Journal, Vol. 2-3, No 1 (1993)*
[2] C. Dumitrescu, V. Seleacu, Some Notions and Questions in Number Theory, *Erhus University Press (1994)*
[3] Sabin and Tatiana Tabirca, Computational Aspects of Smarandache's Function, *Smarandache Notions Journal, Vol. 8, No. 1-2-3, Fall 1997*
[4] Henry Ibstedt, The Florentin Smarandache Function S(n), *Smarandache Function Journal, Vol. 2-3, No 1 (1993)*
[5] C. Dumitrescu, V. Seleacu, The Smarandache Function, *Erhus University Press (1996)*

Chapter IV

Periodic Sequences

IV.1 Introduction

In *Mathematical Spectrum*, vol 29 No 1 [1], is an article on Smarandache's periodic sequences which terminates with the statement:

> *"There will always be a periodic sequence whenever we have a function $f: S \to S$, where S is a finite set of positive integers and we repeat the function f."*

However, consider the following trivial function $f(x_k): S \to S$, where S is an ascending set of integers $\{a_1, a_2, \ldots a_r, \ldots a_n\}$:

$$f(x_k) = \begin{cases} x_{k-1} \text{ if } x_k > a_r \\ x_k \text{ if } x_k = a_r \\ x_{k+1} \text{ if } x_k < a_r \end{cases}$$

As we can see the iteration of the function f in this case converges to an invariant a_r, which we may of course consider as a sequence (or loop) of only one member. In this study, however, a periodic sequence will be referred to as such if it has at least two members. If it has only one member it will be referred to as invariant.

There is one more snag to overcome. In the Smarandache sequences 05 is considered as a two-digit integer. The consequence of this is that 00056 is considered as as a five digit integer while 056 is considered as a three-digit integer. We will abolish this ambiguity, 05 is a one-digit integer and 00200 is a three-digit integer.

With these two remarks in mind let's look at these sequences. There are in all four different ones reported in the above mentioned article in Mathematical Spectrum. The study of the first one will be carried out in much detail in view of the above remarks.

IV.2 The Two-Digit Smarandache Periodic Sequence

It has been assumed that the definition given below leads to a repetition according to Dirichlet's box principle (or the statement made above). However, as we will see, this definition leads to a collapse of the sequence.

Preliminary definition. Let N_k be an integer of at most two digits and let N_k' be its digital reverse. We define the element N_{k+1} of the sequence through

$$N_{k+1} = |N_k - N_k'|$$

where the sequence is initiated by an arbitrary two digit integer N_1.

Let's write N_1 in the form $N_1=10a+b$ where a and b are digits. We then have

$$N_2 = |10a+b-10b-a| = 9 \cdot |a-b|$$

The $|a-b|$ can only assume 10 different values 0,1,2, ... ,9. This means that N_3 is generated from only 10 different values of N_2. Let's first find out which two digit integers result in $|a-b| = 0,1,2,$ .. and 9 respectively.

$\|a-b\|$	Corresponding two digit integers																	
0	11	22	33	44	55	66	77	88	99									
1	10	12	21	23	32	34	43	45	54	56	65	67	76	78	87	89	98	
2	13	20	24	31	35	42	46	53	57	64	68	75	79	86	97			
3	14	25	30	36	41	47	52	58	63	69	74	85	96					
4	15	26	37	40	48	51	59	62	73	84	95							
5	16	27	38	49	50	61	72	83	94									
6	17	28	39	60	71	82	93											
7	19	29	70	81	92													
8	19	80	91															
9	90																	

It is now easy to follow the iteration of the sequence which invariably terminates in 0, table 1.

Table 1. Iteration of sequence according to the preliminary definition

|a-b|	N_2	N_3	N_4	N_5	N_5	N_6
0	0					
1	9	0				
2	18	63	27	45	9	0
3	27	45	9	0		
4	36	27	45	9	0	
5	45	9	0			
6	54	9	0			
7	63	27	45	9	0	
8	72	45	9	0		
9	81	63	27	45	9	0

The termination of the sequence is preceded by the one digit element 9 whose reverse is 9. The following definition is therefore proposed.

Definition of Smarandache's two-digit periodic sequence. Let N_k be an integer of at most two digits. N_k' is defined through

$$N_k' = \begin{cases} \text{the reverse of } N_k \text{ if } N_k \text{ is a two digit integer} \\ N_k \cdot 10 \text{ if } N_k \text{ is a one digit integer} \end{cases}$$

We define the element N_{k+1} of the sequence through

$$N_{k+1} = |N_k - N_k'|$$

where the sequence is initiated by an arbitrary two digit integer N_1 with unequal digits.

Modifying table 1 according to the above definition results in table 2.

Conclusion: The iteration always produces a loop of length 5 which starts on the second or the third term of the sequence. The period is 9, 81, 63, 27, 45 or a cyclic permutation thereof.

Table 2. Iteration of the Smarandache two digit sequence

a-b	N_2	N_3	N_4	N_5	N_5	N_6	N_7
1	9	81	63	27	45	9	
2	18	63	27	45	9	81	63
3	27	45	9	81	63	27	
4	36	27	45	9	81	63	27
5	45	9	81	63	27	45	
6	54	9	81	63	27	45	9
7	63	27	45	9	81	63	
8	72	45	9	81	63	27	45
9	81	63	27	45	9	81	

IV.3 The Smarandache n-digit periodic sequence.

Let's extend the definition of the two-digit periodic sequence in the following way.

Definition of the Smarandache n-digit periodic sequence.

Let N_k be an integer of at most n digits and let R_k be its reverse. N_k' is defined through

$$N_k' = R_k \cdot 10^{n-1-[\log_{10} N_k]}$$

We define the element N_{k+1} of the sequence through

$$N_{k+1} = |N_k - N_k'|$$

where the sequence is initiated by an arbitrary n-digit integer N_1 in the domain $10^n \le N_1 < 10^{n+1}$. It is obvious from the definition that $0 \le N_k < 10^{n+1}$, which is the range of the iterating function.

Let's consider the cases n=3, n=4, n=5 and n=6.

n=3.

Domain $100 \le N_1 \le 999$. . The iteration will lead to an invariant or a loop (periodic sequence)[1]. There are 90 symmetric integers in the domain, 101, 111, 121, ...202, 212, ..., for which $N_2=0$ (invariant). All other initial integers iterate into various entry points of the same periodic sequence. The number of numbers in the domain resulting in each entry of the loop is denoted s in table 3.

Table 3. Smarandache 3-digit periodic sequence

s	239	11	200	240	120
Loop	99	891	693	297	495

It is easy to explain the relation between this loop and the loop found for n=2. Consider $N=a_0+10a_1+100a_2$. From this we have $|N-N'|=99|a_2-a_0|=11 \cdot 9|a_2-a_0|$ which is 11 times the corresponding expression for n=2 and as we can see this produces a 9 as middle (or first) digit in the sequence for n=3.

n=4.

Domain $1000 \le N_1 \le 9999$. The largest number of iterations carried out in order to reach the first member of the loop is 18 and it happened for $N_1=1019$. The iteration process ended up in the invariant 0 for 182 values of N_1, 90 of these are simply the symmetric integers in the domain like $N_1=4334$, 1881, 7777, etc., the other 92 are due to symmetric integers obtained after a couple of iterations. Iterations of the other 8818 integers in the domain result in one of the following 4 loops or a cyclic permutation of one of these. The number of numbers in the domain resulting in each entry of the loops is denoted s in table 4.

[1] This is elaborated in detail in *Surfing on the Ocean of Numbers* by the author, Vail Univ. Press 1997.

Table 4. Smarandache 4-digit periodic sequences

s	378	259			
Loop	2178	6534			
s	324	18	288	2430	310
Loop	90	810	630	270	450
s	446	2	449	333	208
Loop	909	8181	6363	2727	4545
s	329	11	290	2432	311
Loop	999	8991	6993	2997	4995

n=5.

Domain $10000 \leq N_1 \leq 99999$. There are 900 symmetric integers in the domain. 920 integers in the domain iterate into the invariant 0 due to symmetries.

Table 5. Smarandache 5-digit periodic sequences

s	3780	2590			
Loop	21978	65934			
s	3240	180	2880	24300	3100
Loop	990	8910	6930	2970	4950
s	4469	11	4490	3330	2080
Loop	9009	81081	63063	27027	45045
s	3299	101	2900	24320	3110
Loop	9999	89991	69993	29997	49995

n=6.

Domain $100000 \leq N_1 \leq 999999$. There are 900 symmetric integers in the domain. 12767 integers in the domain iterate into the invariant 0 due to symmetries. The longest sequence of iterations before arriving at the first loop member is 53 for N=100720. The last column in table 6 shows the number of integers iterating into each loop.

Table 6. Smarandache 6-digit periodic sequences

s	13667									13667
L 1	0									
s	13542	12551								26093
L 2	13586	65340								
s	12685	12685								26271
L 3	219978	659934								
s	19107	2711	7127	123320	12446					164711
L 4	900	8100	6300	2700	4500					
s	25057	18	12259	20993	4449					62776
L 5	9090	81810	63630	27270	45450					
s	47931	14799	42603	222941	29995					358269
L 6	9990	89910	69930	29970	49950					
s	25375	11	12375	21266	4409					63436
L 7	90009	810081	630063	270027	450045					
s	1488	2	1005	1033	237					3765
L 8	90909	818181	636363	272727	454545					
s	1809	11	1350	1570	510					5250
L 9	99099	891891	693693	297297	495495					
s	19139	2648	7292	123673	12472					165224
L 10	99999	899991	699993	299997	499995					
s	152	4	1254	972	492	111	826	485	429	4725
L 11	10989	978021	857142	615384	131868	736263	373626	252747	494505	
s	623	64	156	796	377	36	525	140	194	
L 12	43659	912681	726462	461835	76329	847341	703593	308286	374517	
ctd	596	117	156	793	327	65	530	139	179	5813
ctd	340956	318087	462726	164538	670923	341847	406296	286308	517374	

Table 7. The subtraction periodic sequence, $10 \le N_1 \le 99$

#1	12	20	1	9	89	97	78	86	67	75	56	64	45	53	34	42	23	31
#2	13	30	2	19	90	8	79	96	68	85	57	74	46	63	35	52	24	41
#3	14	40	3	29	91	18	80	7	69	95	58	84	47	73	35	62	25	51
#4	15	50	4	39	92	28	81	17	70	6	59	94	48	83	37	72	26	61
#5	16	60	5	49	93	38	82	27	71									

IV.4 The Smarandache Subtraction Periodic Sequence

Definition: Let N_k be a positive integer of at most n digits and let R_k be its digital reverse. N_k' is defined through

$$N_k' = R_k \cdot 10^{n-1-[\log_{10} N_k]}$$

We define the element N_{k+1} of the sequence through

$$N_{k+1} = |N_k' - c|$$

where c is a positive integer. The sequence is initiated by an arbitrary positive n-digit integer N_1. It is obvious from the definition that $0 \leq N_k < 10^{n+1}$, which is the range of the iterating function.

$c=1$, $n=2$, $10 \leq N_1 \leq 99$

When N_1 is of the form $11 \cdot k$ or $11 \cdot k-1$ then the iteration process results in 0, see figure 1a.

Every other member of the interval $10 \leq N_1 \leq 99$ is a entry point into one of five different cyclic periodic sequences. Four of these are of length 18 and one of length 9 as shown in table 7 and illustrated in figures 1b and 1c, where important features of the iteration chains are shown.

	99	
	98	
	88	←
← (=)	87	
	77	
→	76	
	66	→(-1)
	65	
	55	
	54	
	44	
	43	
	33	
	32	
	22	
	21	
	11	
	10	
	0	

Fig. 1a

	37	
	72	
	26	
	61	
	15	
	50	
→	04	←
←(-1)	39	→(+9)
	92	
	28	
	81	
	17	
	70	
→	06	←
(-1)←	59	→(+9)
	94	
	48	
	83	
	37	

Fig. 1b

	38	
	82	
	27	
	71	
	16	
	60	
→	05	←
(-1)←	49	→(+9)
	93	
	38	

Fig. 1c

$1 \leq c \leq 9$, $n=2$, $100 \leq N_1 \leq 999$

A computer analysis revealed a number of interesting facts concerning the application of the iterative function.

There are no periodic sequences for c=1, c=2 and c=5. All iterations result in the invariant 0 after, sometimes, a large number of iterations.

Table 8. Loop statistics, L=length of loop, f=first term of loop

c	f↓ / L→	0	11	22	33	50	100	167	189	200
1	N_1	900								
2	N_1	900								
3	N_1	241			59			150		
	1							240		
	2							210		
4	N_1	494				42				
	1					364				
5	N_1	900								
6	N_1	300			59		84			
	1						288			
	2						169			
7	N_1	109								535
	1									101
	2									101
	3									14
	4									14
	5									13
	6									13
8	N_1	203				43	85			
	1						252			
	2					305				
	3						12			
9	N_1	21	79	237					170	
	4								20	
	5								10	
	6		161							
	7			121						
	8			81						

For the other values of c there are always some values of N_1 which do not produce periodic sequences but terminate on 0 instead. Those values of N_1 which produce periodic sequences will either have N_1 as the first term of the sequence or one of the values f determined by $1 \le f \le c-1$ as first term. There are only eight different possible value for the length of the loops, namely 11, 22, 33, 50, 100, 167, 189, 200. Table 8 shows how many of the 900 initiating integers in the interval $100 \le N_1 \le 999$ result in each type of loop or invariant 0 for each value of c.

A few examples:
For c=2 and N_1=202 the sequence ends in the invariant 0 after only 2 iterations:
202 200 **0**
For c=9 and N_1=208 a loop is closed after only 11 iterations:
208 793 388 874 469 955 550 46 631 127 712 **208**
For c=7 and N_1=109 we have an example of the longest loop obtained. It has 200 elements and the loop is closed after 286 iterations:

109 894 491 187 774 470 67 753 350 46 633 329 916 612 209 895 591 188 874 471
167 754 450 47 733 330 26 613 309 896 691 189 974 472 267 755 550 48 833 331
126 614 409 897 791 190 84 473 367 756 650 49 933 332 226 615 509 898 891 191
184 474 467 757 750 50 43 333 326 616 609 899 991 192 284 475 567 758 850 51
143 334 426 617 709 900 2 193 384 476 667 759 950 52 243 335 526 618 809 901
102 194 484 477 767 760 60 53 343 336 626 619 909 902 202 195 584 478 867 761
160 54 443 337 726 620 19 903 302 196 684 479 967 762 260 55 543 338 826 621
119 904 402 197 784 480 77 763 360 56 643 339 926 622 219 905 502 198 884 481
177 764 460 57 743 340 36 623 319 906 602 199 984 482 277 765 560 58 843 341
136 624 419 907 702 200 5 493 387 776 670 69 953 352 246 635 529 918 812 211
105 494 487 777 770 70 63 353 346 636 629 919 912 212 205 495 587 778 870 71
163 354 446 637 729 920 22 213 305 496 687 779 970 72 263 355 546 638 829 921
122 214 405 497 787 780 80 73 363 356 646 639 929 922 222 215 505 498 887 781
180 74 463 357 746 640 39 923 322 216 605 499 987 782 280 75 563 358 846 641
139 924 422 217 705 500 2

IV.5 The Smarandache Multiplication Periodic Sequence

Definition: Let c>1 be a fixed integer and N_0 and arbitrary positive integer. N_{k+1} is derived from N_k by multiplying each digit x of N_k by c retaining only the last digit of the product cx to become the corresponding digit of N_{k+1}.

In this case each digit position goes through a separate development without interference with the surrounding digits. Let's as an example consider the third digit of a 6-digit integer for c=3. The iteration of the third digit follows the schema:

xx7yyy —— the third digit has been arbitrarily chosen to be 7.
xx1yyy
xx3yyy
xx9yyy
xx7yyy —— which closes the loop for the third digit.

Let's now consider all the digits of a six-digit integer 237456:

237456
691258
873654
419852
237456 —— which closes the loop.

The digits 5 and 0 are invariant under this iteration. All other digits have a period of 4 for c=3.

Conclusion: Integers whose digits are all equal to 5 are invariant under the given operation. All other integers iterate into a loop of length 4.

We have seen that the iteration process for each digit for a given value of c completely determines the iteration process for any n-digit integer. It is therefore of interest to see these single digit iteration sequences:

With the help of table 9 it is now easy to characterize the iteration process for each value of c.

Integers composed of the digit 5 result in an invariant after one iteration.
Apart form this we have for:

c=2. Four term loops starting on the first or second term.

c=3. Four term loops starting with the first term.

c=4. Two term loops starting on the first or second term (could be called a switch or pendulum).

Table 9. One-digit multiplication sequences

c=2					
1	2	4	8	6	2
2	4	8	6	2	
3	6	2	4	8	6
4	8	6	2	4	
5	0	0			
6	2	4	8	6	
7	4	8	6	2	4
8	6	2	4	8	
9	8	6	2	4	8

c=3				
1	3	9	7	1
2	6	8	4	2
3	9	7	1	3
4	2	6	8	4
5	5			
6	8	4	2	6
7	1	3	9	7
8	4	2	6	8
9	7	1	3	9

c=4			
1	4	6	4
2	8	2	
3	2	8	2
4	6	4	
5	0	0	
6	4	6	
7	8	2	8
8	2	8	
9	6	4	6

c=5		
1	5	5
2	0	0
3	5	5
4	0	0
5	5	
6	0	0
7	5	5
8	0	0
9	5	5

c=6		
1	6	6
2	2	
3	8	8
4	4	
5	0	0
6	6	
7	2	2
8	8	
9	4	4

c=7				
1	7	9	3	1
2	4	8	6	2
3	1	7	9	3
4	8	6	2	4
5	5			
6	2	4	8	6
7	9	3	1	7
8	6	2	4	8
9	3	1	7	9

c=8					
1	8	4	2	6	8
2	6	8	4	2	
3	4	2	6	8	4
4	2	6	8	4	
5	0	0			
6	8	4	2	6	
7	6	8	4	2	6
8	4	2	6	8	
9	2	6	8	4	2

c=9		
1	9	1
2	8	2
3	7	3
4	6	4
5	5	
6	4	6
7	3	7
8	2	8
9	1	9

c=5. Invariant after one iteration.

c=6. Invariant after one iteration.

c=7. Four term loop starting with the first term.

c=8. Four term loop starting with the second term.

c=9. Two term loops starting with the first term (pendulum).

IV.6 The Smarandache Mixed Composition Periodic Sequence

Definition. Let N_0 be a two-digit integer $a_1 \cdot 10 + a_0$. If $a_1 + a_0 < 10$ then $b_1 = a_1 + a_0$ otherwise $b_1 = a_1 + a_0 + 1$. $b_0 = |a_1 - a_0|$. We define $N_1 = b_1 \cdot 10 + b_0$. N_{k+1} is derived from N_k in the same way.[2]

There are no invariants in this case. 36, 90, 93 and 99 produce two-element loops. The longest loops have 18 elements. A complete list of these periodic sequences is presented below.

10 11 20 22 40 44 80 88 70 77 50 55 10

11 20 22 40 44 80 88 70 77 50 55 10 11

12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12

13 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42

14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14

15 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64

16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16

17 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86

18 97 72 95 54 91 18

19 18 97 72 95 54 91 18

20 22 40 44 80 88 70 77 50 55 10 11 20

21 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31

22 40 44 80 88 70 77 50 55 10 11 20 22

23 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51

24 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62

---

[2] Formulation conveyed to the author: "Let N be a two-digit number. Add the digits, and add them again if the sum is greater than 10. Also take the absolute value of their difference. These are the first and second digits of N_1."

25 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73
26 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84
27 95 54 91 18 97 72 95
28 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16
29 27 95 54 91 18 97 72 95
30 33 60 66 30
31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31
32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32
33 60 66 30 33
34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34
35 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82
36 93 36
37 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14
38 25 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73
39 36 93 36
40 44 80 88 70 77 50 55 10 11 20 22 40
41 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53
42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42
43 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71
44 80 88 70 77 50 55 10 11 20 22 40 44
45 91 18 97 72 95 54 91
46 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12
47 23 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51
48 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34
49 45 91 18 97 72 95 54 91
50 55 10 11 20 22 40 44 80 88 70 77 50
51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51
52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52
53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53
54 91 18 97 72 95 54
55 10 11 20 22 40 44 80 88 70 77 50 55
56 21 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31
57 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32
58 43 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71
59 54 91 18 97 72 95 54
60 66 30 33 60
61 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75
62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62
63 93 36 93
64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64
65 21 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31
66 30 33 60 66
67 41 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53
68 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52
69 63 93 36 93

70 77 50 55 10 11 20 22 40 44 80 88 70
71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71
72 95 54 91 18 97 72
73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73
74 23 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75 32 51
75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75
76 41 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53
77 50 55 10 11 20 22 40 44 80 88 70 77
78 61 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75
79 72 95 54 91 18 97 72
80 88 70 77 50 55 10 11 20 22 40 44 80
81 97 72 95 54 91 18 97
82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82
83 25 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86 52 73
84 34 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84
85 43 71 86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71
86 52 73 14 53 82 16 75 32 51 64 12 31 42 62 84 34 71 86
87 61 75 32 51 64 12 31 42 62 84 34 71 86 52 73 14 53 82 16 75
88 70 77 50 55 10 11 20 22 40 44 80 88
89 81 97 72 95 54 91 18 97
90 99 90
91 18 97 72 95 54 91
92 27 95 54 91 18 97 72 95
93 36 93
94 45 91 18 97 72 95 54 91
95 54 91 18 97 72 95
96 63 93 36 93
97 72 95 54 91 18 97
98 81 97 72 95 54 91 18 97
99 90 99

Chapter V

Smarandache Concatenated Sequences

V.1 Introduction

Smarandache formulated a series of very artificially conceived sequences through concatenation. The sequences studied below are special cases of the Smarandache Concatenated S-sequence.

Definition: Let G={g_1, g_2, ... g_k, ... } be an ordered set of positive integers with a given property G. The corresponding concatenated S.G sequence is defined through

$$S.G = \{a_i : a_1 = g_1, a_k = a_{k-1} \cdot 10^{1+\log_{10} g_k} + g_k, k \geq 1\}$$

In table 1 the first 20 terms are listed for three cases, which we will deal with in some detail below.

V.2 The Smarandache Odd Sequence

The **Smarandache Odd Sequence** is generated by choosing G={1,3,5,7,9,11,.....}. Smarandache asks how many terms in this sequence are primes and as is often the case we have no answer. But for this and the other concatenated sequences we can take a look at a fairly large number of terms and see how frequently we find primes or potential primes. As in the case of prime-product sequence we will resort to Fermat's little theorem to find all primes/pseudo-primes among the first 200 terms. If they are not too big we can then proceed to test if they are primes. For the Smarandache Odd Sequence there are only five cases which all were confirmed to be primes using the elliptic curve prime factorization program. In table 2 # is the term

number, L is the number of digits of N and N is a prime number member of the Smarandache Odd Sequence :

Table 1a. The first 20 terms of the Smarandache Odd Sequence

1
13
135
1357
13579
1357911
135791113
13579111315
1357911131517
135791113151719
13579111315171921
1357911131517192123
135791113151719212325
13579111315171921232527
1357911131517192123252729
135791113151719212325272931
13579111315171921232527293133
1357911131517192123252729313335
135791113151719212325272931333537
13579111315171921232527293133353739
1357911131517192123252729313335373941

Table 2. Prime numbers in the Smarandache Odd Sequence

#	L	N
2	2	13
10	15	135791113151719
16	27	135791113151719212325272931
34	63	135791113151719212325272931333537394143454749515355575961636567
49	93	135791113151719212325272931333537394143454749515355575961636567697173757779818385878991939597

Term #201 is a 548 digit number.

V.3 The Smarandache Even Sequence

The Smarandache Even Sequence is generated by choosing G={2,4,6,8,10, ...}. The question here is : How many terms are *nth* powers of a positive integer?

A term which is a *nth* power must be of the form $2^n \cdot a$ where a is an odd *nth* power. The first step is therefore to find the highest power of 2 which divides a given member of the sequence, i.e. to determine n and at the same time we will find a. We then have to test if a is a *nth* power. The *Ubasic* program below has been implemented for the first 200 terms of the sequence. No *nth* powers were found.

Table 1b. The first 20 terms of the Smarandache Even Sequence

2
24
246
2468
246810
24681012
2468101214
246810121416
24681012141618
2468101214161820
246810121416182022
24681012141618202224
2468101214161820222426
246810121416182022242628
24681012141618202224262830
2468101214161820222426283032
246810121416182022242628303234
24681012141618202224262830323436
2468101214161820222426283032343638

246810121416182022242628303234363840
24681012141618202224262830323436384042

Ubasic program: (only the essential part of the program is listed)

```
60  N=2
70  for U%=4 to 400 step 2
80  D%=int(log(U%)/log(10))+1        'Determine length of U%
90  N=N*10^D%+U%                     'Add on U%
100 A=N:E%=0
110 repeat
120 A1=A:A=A\2:inc E%                'Determine E% (=n)
130 until res<>0
132 dec E%:A=A1                      'Determine A (=a)
140 B=round(A^(1/E%))
150 if B^E%=A then print E%,N        'Check if a is a nth power
160 next
170 end
```

So there is not even a perfect square among the first 200 terms of the Smarandache Even Sequence. Are there terms in this sequence which are 2·p where p is a prime (or pseudo prime). With a small change in the program used for the Smarandache Odd Sequence we can easily find out. Strangely enough not a single term was found to be of the form 2·p.

V.4 The Smarandache Prime Sequence

The **Smarandache Prime Sequence** is generated by {2,3,5,7,11, ...}. Again we ask: - How many are primes? - and again we apply the method of finding the number of primes/pseudo primes among the first 200 terms.

There are only 4 cases to consider: Terms #2 and #4 are primes, namely 23 and 2357. The other two cases are: term #128 which is a 355 digit number and term #174 which is a 499 digit number.

#128
2357111317192329313741434753596167717379838997101103107109113127131137139
1491511571631671731791811911931971992112232272292332392412512572632692712
7728128329330731131331733133734734935335936737337938338939740140941942143

1433439443449457461463467479487491499503509521523541547557563569571577587
593599601607613617619631641643647653659661673677683691701709719

Table 1c. The first 20 terms of the Smarandache Prime Sequence

2
23
235
2357
235711
23571113
2357111317
235711131719
23571113171923
2357111317192329
235711131719232931
23571113171923293137
2357111317192329313741
235711131719232931374143
23571113171923293137414347
2357111317192329313741434753
235711131719232931374143475359
23571113171923293137414347535961
2357111317192329313741434753596167
235711131719232931374143475359616771
23571113171923293137414347535961677173

#174
2357111317192329313741434753596167717379838997101103107109113127131137139
1491511157163167173179181191193197199211223227229233239241251257263269271 2
77281283293307311313317331337347349353359367373379383389397401409419421 43
1433439443449457461463467479487491499503509521523541547557563569571577587
593599601607613617619631641643647653659661673677683691701709719727733739 7
4375175776176977378779780981182182382782983985385785986387788188388790791
1919929937941947953967971977983991997100910131019102110311033

Are these two numbers prime numbers?

Chapter VI

On the Harmonic Series

VI.1 Comparison of a few sequences

The harmonic series $1+\frac{1}{2}+\frac{1}{3}\ \dots\ \frac{1}{k}+\dots$ is so important and so intensely studied that nothing more can be said about it - or can there? We may always pose a few questions.

We know that the series is divergent. Furthermore the series $\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}}$ is convergent for $\alpha>1$ and divergent for $\alpha\leq 1$. Since it is the "borderline" it may be interesting to examine the following questions:

Question 1. What is the smallest number of terms m we need in order to make the sum greater than a given positive integer w? In other words the smallest m for which $\sum_{k=1}^{m}\frac{1}{k}\geq w$.

Question 2. What is the smallest m for which $\sum_{k=1}^{m}\frac{1}{k^{1.02}}\geq w$?

Using the notations for the Smarandache function and the square residues introduced in chapter II.2,3 we pose the corresponding questions:

Question 3. What is the smallest m for which $\sum_{k=1}^{m}\frac{1}{S^2(k)}\geq w$?

Question 4. What is the smallest m for which $\sum_{k=1}^{m}\frac{1}{S_r{}^2(k)}\geq w$?

Table 1 shows the values of m required in each case for w=1, 2, 3, ...12. In diagram 1 ln(m) is plotted against w. A simplified version of this diagram is shown on the cover. We see that the series $\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}}$ is very "sensitive" to

changes in the value of α. $\sum_{k=1}^{m} \frac{1}{S_r^2(k)}$ is on the convergent side of the graph for $1/k$ and $\sum_{k=1}^{m} \frac{1}{S^2(k)}$ on the divergent side.

Table 1. Comparison of sequences.

w	Series 1: $1/k$ Values of m	Series 2: $1/S^2(k)$ Values of m	Series 3: $1/S_r^2(k)$ Values of m	Series 4: $1/k^{1.02}$ Values of m
1	2	24	2	2
2	4	144	9	4
3	11	462	54	12
4	31	1045	243	35
5	83	1944	729	102
6	227	3200	2048	311
7	616	4862	6561	966
8	1674	6912	16384	3082
9	4550	9477	37746	10115
10	12367	12480	93312	34167
11	33617	16008	209952	118971
12	91380	20020	497664	427698

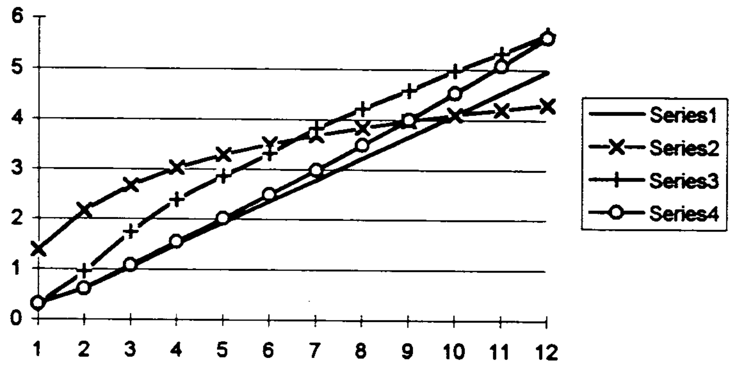


Diagram 1. The behaviour of some sequences: ln(m) plotted against w.

VI.2 Integers represented as sums of terms of the harmonic series

Given a positive integer w. Can we represent w as a sum of a finite number of different terms chosen from the harmonic series? For w=1 we have the simple answer $1=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$, but what about w=2, w=3 etc.

Method 1: k-perfect numbers. By definition a k-perfect number is a number n which satisfies the equation $\sigma(n)=k\cdot n$, where $\sigma(n)$ is the sum of divisors of (1 and n included). $k=\frac{\sigma(n)}{n}$ therefore gives the desired representation of the integer k. The result of a computer search for k-perfect numbers is given in table 2. A column is included for the number of divisors m since this is equal to the number of terms required for the representation. The numbers are given in factor form as this is more interesting than the decimal representation. A 2-perfect number is in the literature simply referred to a perfect number. They have attracted a lot of interest - *The problem of perfect numbers, a favorite with ancient Greeks, owes its origin to the number mysticism of the Pythagoreans* (quote from Elementary Number Theory, Uspensky Heaslet).

Table 2. k-perfect numbers n, m = the number of divisors.

k	m	n
2	4	$2\cdot3=6$
2	6	$2^2\cdot7=28$
2	10	$2^4\cdot31=496$
2	14	$2^6\cdot127=8128$
2	26	$2^{12}\cdot8191=33550336$
2	34	$2^{16}\cdot131071=8589869056$
2	38	$2^{18}\cdot524287=137438691328$
2	62	$2^{30}\cdot2147483647=2305843008139952128$
3	16	$2^3\cdot3\cdot5=120$
3	24	$2^5\cdot3\cdot7=672$
3	288	$2^8\cdot5\cdot7\cdot19\cdot37\cdot73=459818240$
3	80	$2^9\cdot3\cdot11\cdot31=523776$
3	224	$2^{13}\cdot3\cdot11\cdot43\cdot127=1476304896$
3	480	$2^{14}\cdot5\cdot7\cdot19\cdot31\cdot151=51001180160$

Table 2. continued.

k	m	n
4	216	$2^2\cdot3^2\cdot5\cdot7^2\cdot13\cdot19=2178540$
4	96	$2^3\cdot3^2\cdot5\cdot7\cdot13=32760$
4	96	$2^5 3^3\cdot5\cdot7=30240$
4	384	$2^7\cdot3^3\cdot5^2\cdot17\cdot31=45532800$
4	320	$2^9\cdot3^3\cdot5\cdot11\cdot31=23569920$
4	480	$2^9\cdot3^2\cdot7\cdot11\cdot13\cdot31=142990848$
4	1056	$2^{10}\cdot3^3\cdot5^2\cdot23\cdot31\cdot89=43861478400$
4	9984	$2^{25}\cdot3^3\cdot5^2\cdot19\cdot31\cdot683\cdot2731\cdot8191$
5	1920	$2^7\cdot3^4\cdot5\cdot7\cdot11^2\cdot17\cdot19=14182439040$
5	2304	$2^7\cdot3^5\cdot5\cdot7^2\cdot13\cdot17\cdot19=31998395520$
5	5280	$2^{10}\cdot3^4\cdot5\cdot7\cdot11^2\cdot19\cdot23\cdot89$
5	6336	$2^{10}\cdot3^5\cdot5\cdot7^2\cdot13\cdot19\cdot23\cdot89$
5	3456	$2^{11}\cdot3^3\cdot5^2\cdot7^2\cdot13\cdot19\cdot31$
5	9216	$2^{11}\cdot3^5\cdot5^3\cdot7^3\cdot13^3\cdot17$
5	43008	$2^{20}\cdot3^3\cdot5^3\cdot7^3\cdot13^3\cdot17\cdot127\cdot337$
5	709632	$2^{21}\cdot3^6\cdot5^2\cdot7\cdot19\cdot23^2\cdot31\cdot79\cdot89\cdot137\cdot547\cdot683\cdot1093$
5	94208	$2^{22}\cdot3^7\cdot5\cdot7\cdot11\cdot19\cdot41\cdot47\cdot151\cdot197\cdot178481$
6	1658880	$2^{17}\cdot3^4\cdot5^3\cdot7^3\cdot11^2\cdot13^2\cdot19^3\cdot31\cdot37\cdot61\cdot73\cdot181$

For w=3 we have a representation by 16 terms:

$$3=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{20}+\frac{1}{24}+\frac{1}{30}+\frac{1}{40}+\frac{1}{60}+\frac{1}{120}$$

The smallest number of terms required for w=5 is 1920 and for 6 a staggering 1,658,880 terms. For w=7 we have no representation at all. We would first have to find 7-perfect number.

No great results. It's like having used a sledge hammer to kill a mosquito and missed the target. But k-perfect number are interesting in themselves.

Method 2: Trivial Expansions. A term $\frac{1}{k}$ in the harmonic series can always be replaced by two other different terms from the harmonic series: $\frac{1}{k}=\frac{k+1}{k(k+1)}=\frac{1}{k+1}+\frac{1}{k(k+1)}$. Given an expansion of an integer w into terms

of the harmonic series it is possible to construct infinitely many such trivial expansions of w. If however, in a given expansion of w, each term is replaced by two other terms as above and furthermore the replacement process is carried out until all terms in the new representation are different from one another and also from the terms in the original representation then a new representation has been obtained. Obviously such representations for w_1 and w_2 with no common terms can be added together to form a representation of $w=w_1+w_2$.

Example:

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$$

Expand:	k	→	k+1	k(k+1)
	2	→	3	6	Not to be used (repeated)
	3	→	4	12
	6	→	7	42

3 and 6 are repeated and since their expansion gives 4, 12 and 7, 42 we have to continue the expansion:

4	→	5	20
12	→	13	156
7	→	8	56
42	→	43	1806

From this we can write the following representation of 2:

$$2=1+\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\frac{1}{8}+\frac{1}{12}+\frac{1}{13}+\frac{1}{20}+\frac{1}{42}+\frac{1}{43}+\frac{1}{56}+\frac{1}{156}+\frac{1}{1806}$$

Adding our original representation of 1 we get the following representation 3:

$$3=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{12}+\frac{1}{13}+\frac{1}{20}+\frac{1}{42}+\frac{1}{43}+\frac{1}{56}+\frac{1}{156}+\frac{1}{1806}$$

As in the case of 3-perfect numbers we have a representation of 3 with 16 terms. The minimum number of terms for which a representation is possible is 11 as can be seen from table 1. Let's try to get closer to this.

Method 3. Maximum density. For a given value of w table 1 gives the minimum number of terms from the harmonic series required to form a representation. For a given value of w=3 this number is m. We can therefore try to choose the first m-1 terms as close together as possible, i.e. we form

$$s = w - \sum_{k=1}^{m-1} \frac{1}{k}$$

where s is a small fraction which we have to expand in harmonic terms. As before we apply the idea to w=3. We get

$$3 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{15} + \frac{1}{230} + \frac{1}{57960}$$

Bravo! Only 13 terms. We had 16 with the other two methods. But no pleasure lasts forever. The nasty little fraction s will give no end to problem when we try large values for w. Although small s will have large numerators and denominators as we shall soon see. Maybe better get the sledge hammer out again and go in search for the 7-perfect number.

VI.3 Partial sums of the harmonic series as rational numbers.

We consider consecutive partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k}$ expressed as rational numbers reduced to lowest terms.

Question 1. Can there be a repetition of numerators?

Assume $S_{n-1} = \frac{a}{b}$ and $S_n = \frac{a}{b_1}$ where (a,b)=1 and $(a,b_1)=1$. We then have $\frac{a}{b_1} = \frac{a}{b} + \frac{1}{n}$ which is equivalent to $an(b-b_1)$, but $(a,bb_1)=1$ and the latter equation is therefore impossible and the assumption is wrong. We conclude:

Two consecutive sums S_{n-1} and S_n cannot have equal numerators when reduced to lowest terms.

A necessary condition for fidig two sums $S_n = \frac{a_n}{b_n}$ and $S_m = \frac{a_m}{b_m}$, m>n+1 with equal numerators $a_n=a_m$ when in lowest terms is therefore that n_1 exists so that $n<n_1\leq m$ and $a_{n_1} < a_{n_1-1}$. Such numerators exist. Examples:

```
a6=49
a7=137
a18=14274301
a17=4214223
```

a_{99}=3609687032357116542338926129882501631572O7
a_{100}=14466636279520351160221518043104131447711
etc

Now, let p be a prime then $a_p>a_{p-1}$ because S_n is a monotonously increasing function of n and

$$S_p=\frac{a_{p-1}}{b_{p-1}}+\frac{1}{p}=\frac{a_{p-1}p+b_{p-1}}{b_{p-1}p}$$

cannot be reduced to lower terms.

Conjecture: Let p_j and p_{j+1} denote consecutive primes. The numerators in the interval $p_j \le n \le p_{j+1}$, after reducing to lowest terms, are all greater than $a_{p_j}-1$, i.e. when a term of the type $\frac{1}{p}$ is added the numerator takes a leap to a higher value from which, although it may decrease, it will never drop to a value below the value before the leap.

Question 2. What is the most frequently occurring denominator in the sequence $S_n=1+\frac{1}{2}+\frac{1}{3}+\ \dots\ +\frac{1}{n}$ when reduced to lowest terms?

Let $S_{n-1}=\frac{a}{b}$ where (a,b)=1. Then $S_n=\frac{a}{b}+\frac{1}{n}$; $S_n=\frac{an+b}{bn}$. Consider the greatest common divisor of b and n, d=(b,n). Then it is seen immediately that the fraction can be reduced by d. Put $b=b_1d$ and $n=n_1d$ and consequently

$$S_n=\frac{an_1+b_1}{b_1n_1d}$$

If a further reduction is possible then it must be by a factor in d or by d itself because $(n_1,b_1)=1$ and $(b_1,a)=1$.

If n=p is a prime number then (b,n)=1 and the denominator os S_n will have p as a factor. The denominator of S_n is therefore unequal to all previous denominators and is at most the first in a series of equal denominators. Sums with equal denominators must therefore be contained between

$$S_{p_j}=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p_j} \text{ and } S_{p_{j+1}}=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p_{j+1}}$$

where p_j and p_{j+1} are consecutive primes. In table 3 the number of equal denominators is denoted q and the corresponding number of digits of the denominator L.

Table 3. Equal denominators for n<1000.

p_j	$p_{j+1}-1$	q	L
89	96	8	39
139	148	10	62
317	330	14	140
891	906	16	389

I hope this book has been more meaningful than reading the number below which is the 389-digit denominator which occurs 16 times in consecutive partial sums of the harmonic series for n<1000.

4179982336319706196962068134998944512773598562712
1827499073959690180767635682039557104883261454628
9717289725066739371544332075631263681868936793360
4957159732797690125645682733080622438318699474351
6944805893417522011858358480060385932366764886574
4940462245050719348300672374671465097882430105554
5638386382683543980298959618721940328647837326571
6039813081070459435310984219943740583597120000
